INVERTED PENDULUM SYSTEM

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I dedicate this to both of my parents, family, lecturers, friends and the electronic engineering education.

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ABSTRACT

Inverted Pendulum System is a physical device consisting in a cylindrical bar which is usually made of aluminium, free to oscillate around a fixed pivot. The pivot is mounted on a cart, which can only moves in a horizontal direction. The cart can be exerted by a variable force. The bar would naturally tend to fall down from the top vertical position, which is a position of unsteady equilibrium. The goal of this project is to stabilize the inverted pendulum from falling down. This is possible by exerting on the cart through the motor a force which tends to contrast the 'free' pendulum dynamics. The correct force has to be calculated measuring the instant values of the horizontal position and the pendulum's angle. There is two controllers used and compared which are the PID controller and LQR controller. The modelling and analysis is done for both of these controllers in MATLAB and Simulink. After that, the performances of these controllers are analyzed especially on the time taken to stabilize the pendulum and also the cart position. Therefore, the best control approach needs to be discovered in order to give the best performance for the inverted pendulum system.

ABSTRAK

'Inverted Pendulum System' adalah suatu alat yang terdiri daripada rod silinder yang selalunya diperbuat daripada aluminium, bebas untuk berpusing di atas suatu paksi yang tetap. Paksi tersebut terletak di atas kereta yang mana hanya boleh bergerak dalam arah melintang. Kereta ini mempunyai motor yang boleh dikenakan daya yang berubah-ubah ke atasnya. Rod tersebut akan cenderung untuk jatuh ke bawah daripada kedudukan menegak di mana ia adalah kedudukan yang tidak mencapai keseimbangan. Tujuan projek ini adalah untuk mengimbangkan bandul (pendulum) tersebut daripada jatuh. Ini boleh terjadi dengan mengenakan daya pada kereta tersebut melalui motor di mana daya ini berlawanan dengan dinamik bandul (pendulum). Daya yang betul dan sesuai hendaklah dikira dengan mengambil kira tentang nilai kedudukan melintang kereta tersebut serta sudut bandul (pendulum) dari paksi menegak. Terdapat dua jenis pengawal (controller) yang digunakan iaitu 'PID controller' dan 'LQR controller'. Pemodelan dan analisis ke atas kedua-dua controller ini dilakukan dalam MATLAB dan Simulink. Kemudian, persembahan kedua-dua pengawal ini dianalisis terutama sekali terhadap masa yang diambil untuk menstabilkan bandul dan juga kedudukan kereta. Oleh itu, pengawal (controller) yang paling baik dan sesuai perlu dicari untuk menghasilkan persembahan yang terbaik untuk 'Inverted Pendulum System'.

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LIST OF ABBREVIATIONS

P	-	Proportional
I	-	Integral
D	-	Derivative
PI	-	Proportional-plus-Integral
PD	-	Proportional-plus- Derivative
PID	-	Proportional-plus-Integral-plus- Derivative
LQG	-	Logic Quadratic Gaussian
e.g.	-	for example
LQR	-	Linear Quadratic Regulator

LIST OF APPENDICES

CHAPTER I

INTRODUCTION

In this chapter, the background of the project study, the objective, as well as the scope of the project and so on will be discussed.

1.1 Project Overview

An inverted pendulum is a physical device consisting in a cylindrical bar (usually of aluminium) free to oscillate around a fixed pivot. The pivot is mounted on a carriage, which in its turn can move on a horizontal direction. The carriage is driven by a motor, which can exert on it a variable force. The bar would naturally tend to fall down from the top vertical position, which is a position of unsteady equilibrium.

The goal of the experiment is to stabilize the pendulum (bar) on the top vertical position. This is possible by exerting on the carriage through the motor a force which tends to contrast the 'free' pendulum dynamics. The correct force has to be calculated measuring the instant values of the horizontal position and the pendulum angle.

The inverted pendulum is a traditional example (neither difficult nor trivial) of a controlled system. Thus it is used in simulations and experiments to show the performance of different controllers (e.g. PID controllers, Linear Quadratic (LQ) control, state space controllers, fuzzy controllers and so on).

At the end of this project, the expected result is an appropriate and the best controller can be found to balance the inverted pendulum system and reduce or solve all the system's problem. The result can be showed by using MATLAB and Simulink and the best controller will be chosen based on the comparison made.

1.2 Problem Statements

Control of inverted pendulum system is recognized as a benchmark problem for various controller designs and widely employed within laboratories for education and research purposes. It is popular due to its simple setup and several interesting features such as the instability, nonlinearity and inherent non-minimum phase characteristics.

The inverted pendulum system is an intriguing subject from the control point of view due to their intrinsic nonlinearity. The problem is to balance a pole on a mobile platform that can move in only two directions, to the left or to the right.

1.3 Objectives

The objectives of this project are to balance the pendulum on a moving cart and to find a better control approach for the Inverted Pendulum System by comparing two different controllers.

1.4 Scope of Project

The scope of this project is to study the concept of an inverted pendulum system. After that, a mathematical model of the inverted pendulum system can be derived. Then, a mathematical model of PID controller for the inverted pendulum system have to be studied and derived before implementing and analyzing the project in MATLAB and Simulink.

1.5 Research Methodology

This project is about the nonlinear system. In that case, the first thing to be learned is the concept of the nonlinear system. After that, the research should be focusing on the concept of inverted pendulum system without any controller. Next is to study how to derive the mathematical model of the inverted pendulum system without any controller.

This nonlinear system needs a controller to control the system. The first controller suggested is the Proportional-Integral-Derivative (PID) controller. Therefore, the concept and how to derive the mathematical model of the inverted pendulum system with PID controller need to be learnt and understand.

All of these derivations will be implemented in MATLAB and Simulink program and the results need to be analyzed. For that reason, it is necessary to learnt and really understand how to use the MATLAB commands and tools. It is really important to be expert in MATLAB and Simulink programming as this project totally employs this MATLAB and Simulink program.

After all of the steps above have been done, the next controller will be given and the same steps will be repeated. Subsequently, the performances of both controllers are compared and the best controller among the two can be concluded.

1.6 Thesis Structure

This thesis consists of five chapters that will explain more details about this project. The first chapter will put in plain words about the project overview which are about the project background, problem statements, objectives, scopes and research methodology.

The second chapter will review about various controllers designed for the inverted pendulum system. From this literature review, different approaches of controller design for the inverted pendulum system can be studied. Furthermore, there are various types of inverted pendulum can be modelled. Therefore, different ways of designing the controllers for the specific inverted pendulum can be learnt.

The third chapter is about the inverted pendulum system concept, mathematical modelling of the inverted pendulum system without controller, PID concept and its mathematical modelling for the inverted pendulum system. This chapter will discuss in details on the research methodology for this project.

The fourth chapter will discuss about the results for this inverted pendulum system project. In addition, the results will be analyzed and the way of solving the problems occurred during this project will be presented.

The fifth chapter which is also the last chapter will discuss about the discussion, recommendation and the conclusion of the project. The recommendation suggested is proposed with the intention of improving the project in future.

CHAPTER II

LITERATURE REVIEW

In this chapter, discussion about the basic concept of inverted pendulum system and the other various controllers used for inverted pendulum system will be presented. It will consist of detail explanation about the model, control approach, results and analysis.

2.1 Literature Review

There are a lot of design methods and difference controller used in designing a suitable controller for the inverted pendulum system.

[1] has presented the Switching Control for Inverted Pendulum System Based on Energy Modification. It is based on a modified energy such that the potential energy is minimums in the upright position. This method of controlling the inverted pendulum system has been derived by paying attention to the energy of pendulum. The model of this inverted pendulum is proposed where there is a mass situated at the centre of the pendulum's pole and the cart dose not has wheels. The equation of motion is derived by concerning the moment of inertia with respect to the pivot point, the mass of the pole, the distance from the pivot to the centre of mass, the acceleration of gravity, the acceleration of the pivot and the angle between the vertical and the pendulum.

After considering the system with damping and how to control a system with damping is figured out, a simulation is performed and the switching conditions were lead and the control parameters were shown.

In [2], inverted pendulum system with the personified intelligent control which is not based on the accurate mathematical model is presented. It used a method that can develop control laws directly by means of qualitative analysis and synthesis of the plant. The inverted pendulum system is a multi-input single-output control system consisting of four inputs; pole angle, change of the pole angle, cart position and change of cart position and single output; control action. The prerequisite of applying this control is to understand the physical structure and behaviour of the controlled object as fully as possible. The model for this control is the inverted cart-pendulum situated on a rail and driven by a single motor. The analysis done by reduction of primary problem until small problems that can be solve. Then, the equations obtained will be programmed in C language and the output can be seen from the graph obtained.

[3] has presented that, the fuzzy logic controller for an inverted pendulum system does not require a precise mathematical modelling of the system nor complex computations. Fuzzy control provides an easy solution to this problem as it is shown by the derivation of the fuzzy linguistic rules and its function verified by computer simulations. From this model of inverted pendulum system, the pendulum's cart receives an external force for balancing the pendulum provided by a permanent magnet servo-motor mounted on the cart. The fuzzy logic controller changes the human language to fuzzy language before it makes decision to control the system and back to human language for further control action.

[4] has designed a nonlinear H_∞ controller for the inverted pendulum system by focusing on how to find a solution for the Hamilton-Jacobi equation. A successive algorithm is employed to obtain an approximate solution of the Hamilton-Jacobi equation and then a nonlinear and the linear H_∞ controllers will be performed and compared. The nonlinear H_∞ controller has better performance and robustness that the

linear controller. The model of the inverted pendulum system in this paper is a normal model which is an inverted pendulum on a moving cart where the force applied. Then the computer simulations for the closed loop system were performed and the results obtained for further analysis.

[5] has used Hybrid LQG-Neural controller for inverted pendulum system where benefits of a genetically optimized neural controller and an LQG controller is combined in a single system controller. High quality of the regulation process is achieved through utilization of the neural controller, while stability of the system during transient processes and a wide range of operation are assured through application of the LQG controller. This controller is validated by applying it to a simulation model of an inverted pendulum system. For the analysis, a test bed and an experiment is done for the development of the hybrid controller. The model of inverted pendulum system is a single inverted pendulum system mounted on a moving cart. The results from simulations are shown in histograms. From this hybrid controller, it has shown certain advantages over the conventional LQG controller.

In [6], A Rule-Based Neural Controller for inverted pendulum system is presented. It demonstrates how a heuristic neural control approach can be used to solve a complex nonlinear control problem. As well as to swing up the pendulum, the controller is also required to bring the cart back to the origin of the track. Specializing to the pendulum problem, the global control task is decomposed into sub-tasks, namely, pendulum positioning and cart positioning. Accordingly, three separate neural sub-controllers are designed to cater to the sub-tasks and their coordination. The simulation result is provided to show the actual performance of the controller. The advantage of this controller is it is able to implement dynamical decisions and rules than just the static mapping actions. The simulation and analysis of the neural controller for the inverted pendulum system is done on a DECstation using the software package Matlab.

2.2 Introduction to Inverted Pendulum System

In our daily life, at some time we may have tried to balance a brush, pen or other object on your index finger or the palm of our hand. We have to constantly adjust the position of the hand to keep the object upright. An inverted pendulum does basically the same thing. However, it is limited in that it only moves in one dimension, despite the fact that our hand could move up, down, sideways and other directions.

Just as balancing with our hand, an inverted pendulum is an inherently unstable system. Force must be properly applied to keep the system intact. To achieve this, proper control theory is required. The inverted pendulum is an invaluable tool for the effective evaluation and comparison various control theories.

An inverted pendulum system consists of a thin rod (pole) attached at its bottom of a moving cart. In a normal condition, a pendulum is stable when hanging downwards, a vertical inverted pendulum is naturally unstable, and must be actively balanced in order to remain upright, usually by moving the cart horizontally as part of a feedback system.

The inverted pendulum system is essential in the evaluation and comparison of various control theories. The inverted pendulum is used in simulations and experiments to show the performance of different controllers (e.g. PID, State Space and Fuzzy Controllers and else).

The inverted pendulum is a classic problem in dynamics and control theory and widely used as benchmark for testing control algorithms (PID controllers, neural networks, genetic algorithms and so on). Variations on this problem include multiple links, allowing the motion of the cart to be commanded while maintaining the pendulum, and balancing the cart-pendulum system. The inverted pendulum is related to rocket or missile guidance, where thrust is actuated at the bottom of a tall vehicle.

Another way that an inverted pendulum may be stabilized, without any feedback or control mechanism, is by oscillating the support rapidly up and down. If the oscillation is sufficiently strong in terms of its acceleration and amplitude, then the inverted pendulum can recover from perturbations in a strikingly counterintuitive manner.

There are many types of models for the inverted pendulum system. Some of them are two-stage of inverted pendulum system, parallel inverted pendulum system, rotational inverted pendulum system, bi-axial inverted pendulum system and many more. In that case, there are also lots of controls approaches have been designed for these various kinds of inverted pendulum system.

In this project, a single inverted pendulum on a moving cart is chosen for designing the control approaches. The model is shown in Figure 2.1.

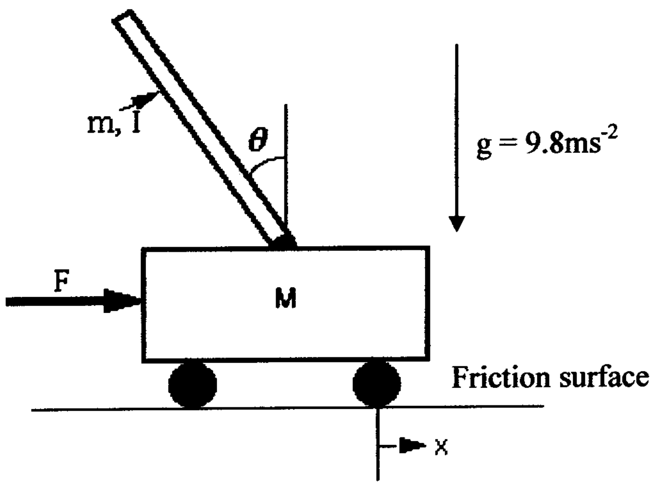

Figure 2.1: Inverted pendulum system on a moving cart

2.3 Applications of Inverted Pendulum

Some considerable applications of inverted pendulum are:

- SIMULATION OF DYNAMICS OF A ROBOTIC ARM

The Inverted Pendulum problem resembles the control systems that exist in robotic arms. The dynamics of Inverted Pendulum simulates the dynamics of robotic arm in the condition when the centre of pressure lies below the centre of gravity for the arm so that the system is also unstable. Robotic arm behaves very much like Inverted Pendulum under this condition.

- MODEL OF A HUMAN STANDING STILL

The ability to maintain stability while standing straight is of great importance for the daily activities of people. The central nervous system (CNS) registers the pose and changes in the pose of the human body, and activates muscles in order to maintain balance. The inverted pendulum is widely accepted as an adequate model of a human standing still (quiet standing).

An inverted pendulum (assuming no attached springs) is unstable, and it is hence obvious that feedback of the state of the pendulum is needed to stabilize the pendulum. Two models for the CNS feedback control are generally considered:

- Time invariant, linear feedback control.
- Linear feedback outside a threshold. No sensory feedback within the threshold.

There are certain passive mechanisms, such as stiffness in muscles and supportive tissue, which may be modelled as a spring and damper.

2.4 PID Controller

PID controller is a common feedback loop component in industrial control system. This kind of controller compares a measured value from process with a reference set point value. The difference or error signal is then used to calculate a new value for a manipulatable input to the process that brings the process measured value back to its desired set point.

Unlike other simpler control algorithms, the PID controller can adjust process outputs based on the history and rate of change of the error signal, which gives more accurate and stable control in cases where a simple proportional control would either have a steady-state error or would cause the process to oscillate. PID controller do not require advanced mathematics to design and can be easily tuned to the desired application, dissimilar with the more complicated control algorithms based on optimal control theory.

A control loop consists of three parts:

a) Measurement by a sensor connected to the process or plant
b) Decision in a controller element
c) Action through an output device

As the controller reads a sensor, it subtracts this measurement from the set point to determine the error. It then uses the error to calculate a correction to the process's input variable so that this correction will remove the error from the process's output measurement.

In a PID loop, correction is calculated from the error in three ways:

a) Cancel out the current error directly (Proportional)
b) The amount of time the error has continued uncorrected (Integral)
c) Anticipate the future error from the rate of change of the error over time (Derivative)

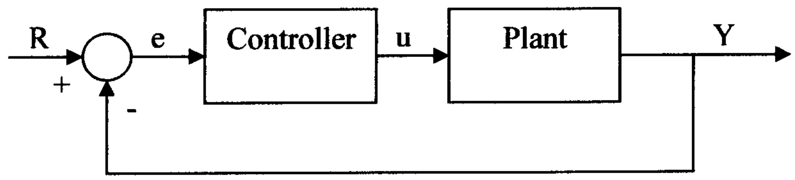

Figure 2.2: A unity feedback system

This is a block diagram of a unity feedback system where the plant is a system to be controlled and the controller provides the excitation for the plant and it is designed to control the overall system behaviour.

The transfer function of the PID controller is:

$$K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s} \tag{2.1}$$

Where K_P is the proportional gain, K_I is the integral gain and K_D is the derivative gain. The variable (e) represents the tracking error, the difference between the desired input value (R) and the actual output (Y). This error signal will be sent to the PID controller and the controller computes both of the derivative and the integral of this error signal. The signal (u) just past the controller is now equal to the proportional gain times the magnitude of the error plus the integral gain times the integral of the error plus the derivative gain times the derivative of the error.

$$u = K_P e + K_I \int e dt + K_D \frac{de}{dt} \tag{2.2}$$

This signal (u) will be sent to the plant and the new output (Y) will be obtained. This new output will be sent back to the sensor again to find the new error signal (e). The controller takes this new error signal and computes its derivative and its integral again. This process goes on and on.

2.4.1 The characteristics of P, I and D controllers

A proportional controller (K_P) will have the effect of reducing the rise time and will reduce, but never eliminate the steady-state error.

An integral control (K_I) will have the effect of eliminating the steady-state error. However, it may make the transient response worse.

A derivative control (K_D) will have the effect of increasing the stability of the system, reducing the overshoot and improving the transient response. Effects on each of controllers K_P, K_I and K_D on a closed loop system are summarized in the table below:

CLOSED LOOP RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	STEADY-STATE ERROR
K_P	Decrease	Increase	Small change	Decrease
K_I	Decrease	Increase	Increase	Eliminate
K_D	Small change	Decrease	Decrease	Small change

Table 2.1: Comparison between K_P, K_I and K_D

The correlations may not be exactly accurate because each gain is dependent of each other. In fact, changing on these variables can change the effect of the other two. For this reason, the table above should only be used as a reference when the user needs to determine the values of K_P, K_I and K_D.

2.4.2 PI and PD Controller

The PI controller consists of an integrator with a proportional gain. This controller will produce a second order response for a first order plant. There will be zero steady-state errors if the reference and disturbance inputs are either unchanging or have step changes.

From the PD controller, it can be seen that the derivative controller (K_D) reduces both the overshoot and the settling time. For the PI controller, it is clear that the integral controller (K_I) decreases the rise time, increases both the overshoot and the settling time and eliminates the steady-state error.

2.4.3 PID Controller Design

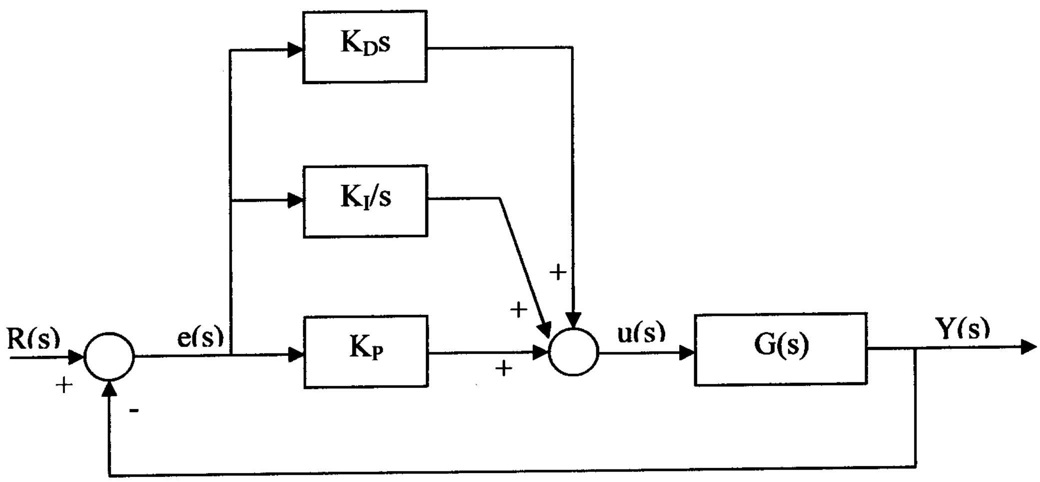

Figure 2.3: A PID controlled system

The design technique for the PID controller consists of the following steps:

a) Evaluate the performance of the uncompensated system to determine how much improvement in transient response is required.

b) Design the PD controller to meet the transient response specifications. The design includes the zero location and the loop gain.

c) Simulate the system to be sure all requirements have been met.

d) Redesign if the simulation shows that requirements have not been met.

e) Design the PI controller to yield the required steady-state error.

f) Determines the gains, K_P, K_I and K_D.

g) Simulate the system to be sure all requirements have been met.

h) Redesign if simulation shows that the requirements have not been met.

2.5 Linear Quadratic Regulator Controller (LQR)

In optimal control one attempt to find a controller that provides the best possible performance with respect to some given measure of performance. For example, the controller that uses the least amount of control-signal energy to take the output to zero. In this case the measure of performance (also called the optimality criterion) would be the control-signal energy.

In general, optimality with respect to some criterion is not the only desirable property for a controller. One would also like stability of the closed-loop system, good gain and phase margins, robustness with respect to unmodelled dynamics and so on.

The theory of optimal control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic functional is called the LQ problem. One of the main results in the theory is that the solution is provided by the linear-quadratic regulator (LQR).

LQR (Linear Quadratic Regulator) is a linear feedback controller that requires a linear system model around an equilibrium point. The optimal LQR is possibly the most important result of modern control and widely used technique of linear control systems.

The LRQ is a controller for state-variable feedback in such a way that

$$u = - Kx$$

so that the value of K is obtained from a minimizing problem of the functional cost,

$$J = \int_0^{\infty} (x'Qx + u'Ru)dt$$

Matrixes Q and R penalize the state error and the control effort, respectively.

The design of the optimal LQR is to find the control gain, K which minimizes the performance index. The behaviour of the optimal LQ controller is determined by the choices of two weighting matrices Q and R within the performance index. However, it is not a trivial problem to decide them.

One must carry out a trial-and-error process to choose the proper weighting matrices in general. One way to begin the process is to use weights that normalize the variables with respect to their largest permissible. Obviously this method does not even guarantee that the specification can be met, even by the optimal system.

CHAPTER III

PROJECT METHODOLOGY

In this chapter, the project methodology will be elaborated comprehensively. All calculations to obtain the mathematical model will be demonstrated step by step.

3.1 Model of Inverted Pendulum System

There are a lot of models for inverted pendulum system. For this project, the descriptions of the model were explained in depth in the next subtopic.

3.1.1 Model Description

An inverted pendulum of mass, m and length, l moves in the vertical plane, about a horizontal axis fixed on a cart with an angle, θ. The cart of mass, M move horizontally in one dimension under the influence of a force, F.

The pendulum rod is assumed to have zero mass. There is a friction, b in the system. The force, F is to be manipulated to keep the pendulum vertical. The input is the force, F and the outputs are the angle, θ and the distance, x.

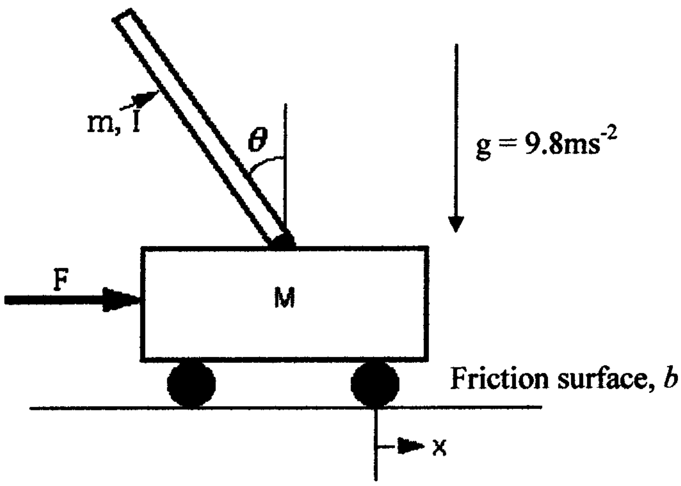

Figure 3.1: Model of Inverted Pendulum System

3.1.2 Inverted Pendulum Parameter

Mass of the cart, $M = 2.0$kg

Mass of the pendulum, $m = 1.0$kg

Length to pendulum center of mass, $L = 1.0$m

Gravitational acceleration, $g = 9.8\text{ms}^{-2}$

Friction of the cart, $b = 0.1$N/m/s

Inertia of the pendulum, $I = 0.006\text{kgm}^{-2}$

Force applied to the cart, F

Cart position coordinate, x

Pendulum angle from vertical, θ

Input = the force, F

Outputs = the angle, θ and the distance, x

3.2 PID Controller for Inverted Pendulum System

3.2.1 Design Requirements

The cart with an inverted pendulum, shown below, is "bumped" with a force, F. The dynamic equations of motion for the system are determined and linearized about the pendulum's angle, theta = 0. In other words, the pendulum is assumed that it does not move more than a few degrees away from the vertical, chosen to be at an angle of 0. The target is to find a controller to satisfy all of the design requirements as below:

The design requirements for this system are:

- Settling time of less than 5 seconds.
- Pendulum angle never more than 0.05 radians from the vertical.

In this project, a PID controller is implemented which it can only be applied to a single-input-single-output (SISO) system. In that case, only the control of the pendulum's angle is highlighted. Therefore, none of the design criteria deal with the cart's position. The system is assumed to start at equilibrium, and experiences an input force. The pendulum should return to its upright position within 5 seconds, and never move more than 0.05 radians away from the vertical.

3.2.2 Force Analysis and System Equations

A free body diagram is a pictorial representation often used by physicists and engineers to analyze the forces acting on a free body. It shows all contact and non-contact forces acting on the body.

Drawing such a diagram can aid in solving for the unknown forces or the equations of motion of the body. Creating a free body diagram can make it easier to understand the forces, and moments, in relation to one another and suggest the proper concepts to apply in order to find the solution to a problem.

The diagrams are also used as a conceptual device to help identify the internal forces, for example shear forces and bending moments in beams, which are developed within structures.

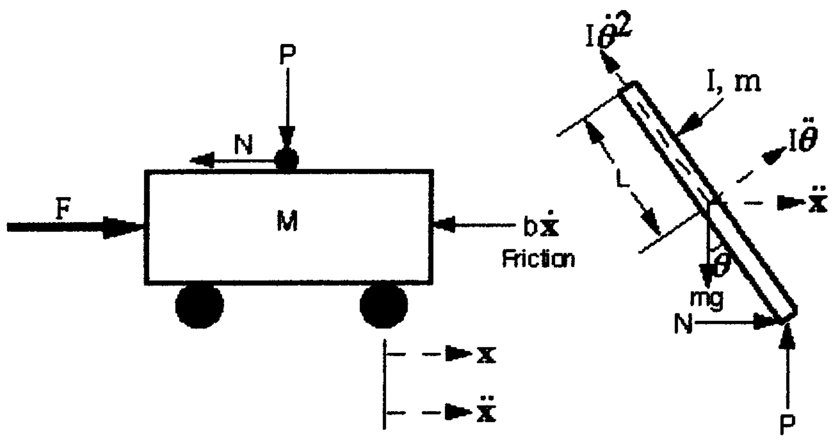

Figure 3.2: Free body diagrams for inverted pendulum system

According to Figure 3.2, a cart of mass M slides on a horizontal track that has friction, b and is pushed by a horizontal force, F. On the cart, an inverted pendulum of mass, *m* is attached via a pivot or pin joint. The pendulum's center of mass is located at a distance, *L* from its two ends, and the pendulum's moment of inertia about its center of mass is denoted by, I. The angle, θ is the angle that the pendulum makes with respect to the vertical axis. The vertical force exerted by the cart on the base of the pendulum is denoted by P, and the horizontal force by N.

This system is tricky to model in Simulink because of the physical constraint which is the pin joint located between the cart and pendulum. It reduces the degrees of freedom in the system. Both the cart and the pendulum have one degree of freedom, *x* and theta respectively. The Newton's equation for these two degrees of freedom will be modelled like below.

$$\frac{d^2x}{dt^2} = \frac{1}{M}\sum_{cart} F_x = \frac{1}{M}\left(F - N - b\frac{dx}{dt}\right)$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{I}\sum_{pendulum} \tau = \frac{1}{I}(NL\cos(\theta) + PL\sin(\theta))$$

It is necessary, however, to include the interaction forces N and P between the cart and the pendulum in order to model the dynamics. The inclusion of these forces requires modelling the x and y dynamics of the pendulum in addition to its theta dynamics.

The interaction forces can be solved manually to obtain the algebra. However, in order to exploit the modelling power of Simulink, the algebra will be taken care by the simulation. Therefore, the additional x and y equations for the pendulum is modelled as below.

$$m\frac{d^2x_p}{dt^2} = \sum_{pendulum} F_x = N$$

$$N = m\frac{d^2x_p}{dt^2}$$

$$m\frac{d^2y_p}{dt^2} = \sum_{pendulum} F_y = P - mg$$

$$P = m\left(\frac{d^2y_p}{dt^2} + g\right)$$

However, x_p and y_p are exact functions of theta. Therefore, their derivatives can be represented in terms of the derivatives of theta.

$$x_p = x - L\sin(\theta)$$

$$\frac{dx_p}{dt} = \frac{dx}{dt} - L\cos\theta\frac{d\theta}{dt}$$

$$\frac{d^2x_p}{dt^2} = \frac{d^2x}{dt^2} + L\sin\theta\left(\frac{d\theta}{dt}\right)^2 - L\cos\theta\frac{d^2\theta}{dt^2}$$

$$y_p = L\cos(\theta)$$

$$\frac{dy_p}{dt} = -L\sin\theta\frac{d\theta}{dt}$$

$$\frac{d^2y_p}{dt^2} = -L\cos\theta\left(\frac{d\theta}{dt}\right)^2 - L\sin\theta\frac{d^2\theta}{dt^2}$$

These expressions can then be substituted into the expressions for N and P. However, rather than continuing with algebra here, these equations will be simply represented in Simulink. This is because Simulink can work directly with nonlinear equations, so it is unnecessary to linearize these equations.

3.2.3 Modelling in Simulink

From the force analysis and system equation setup before, the information will be used in modelling the system in Simulink. Initially, the states of the system needed to be modelled in theta and x where the Newton's equations for the pendulum rotational inertia and the cart mass are represented. In order to construct the equations in Simulink, all of the steps are explained below:

- A new model window in Simulink is opened and resized to give plenty of room as this is a large model.
- Two integrators are inserted from the Linear block library near the bottom of the model and they are connected in series.
- A line is drawn from the second integrator and labelled as "theta" by double-clicking on where the label has to be.
- The line connecting the integrators is labelled as "theta_dot".
- A line is drawn leading to the first integrator and labelled as "theta_doubledot".
- A Gain block is inserted from the Linear block library to the left of the first integrator and its output is connected to the "theta_doubledot" line.
- The gain value of this block is edited by double clicking it and changed to "1/I".
- The label of this block is changed to "pendulum inertia" by clicking on the word "Gain".
- A Sum block is inserted from the Linear block library to the left of the Pendulum Inertia block and its output is connected to the inertia's input.
- The label of this block is changed to "sum torques on pendulum".
- A similar set of elements is constructed near the top of the model with the signals labelled with "x" rather than "theta". The gain block should have the value "1/M" with the label "cart mass", and the Sum block should have the label "sum forces on cart".
- The Sum Forces block is edited and its signs are changed to "-+-" where it represents the signs of the three horizontal forces acting on the cart.

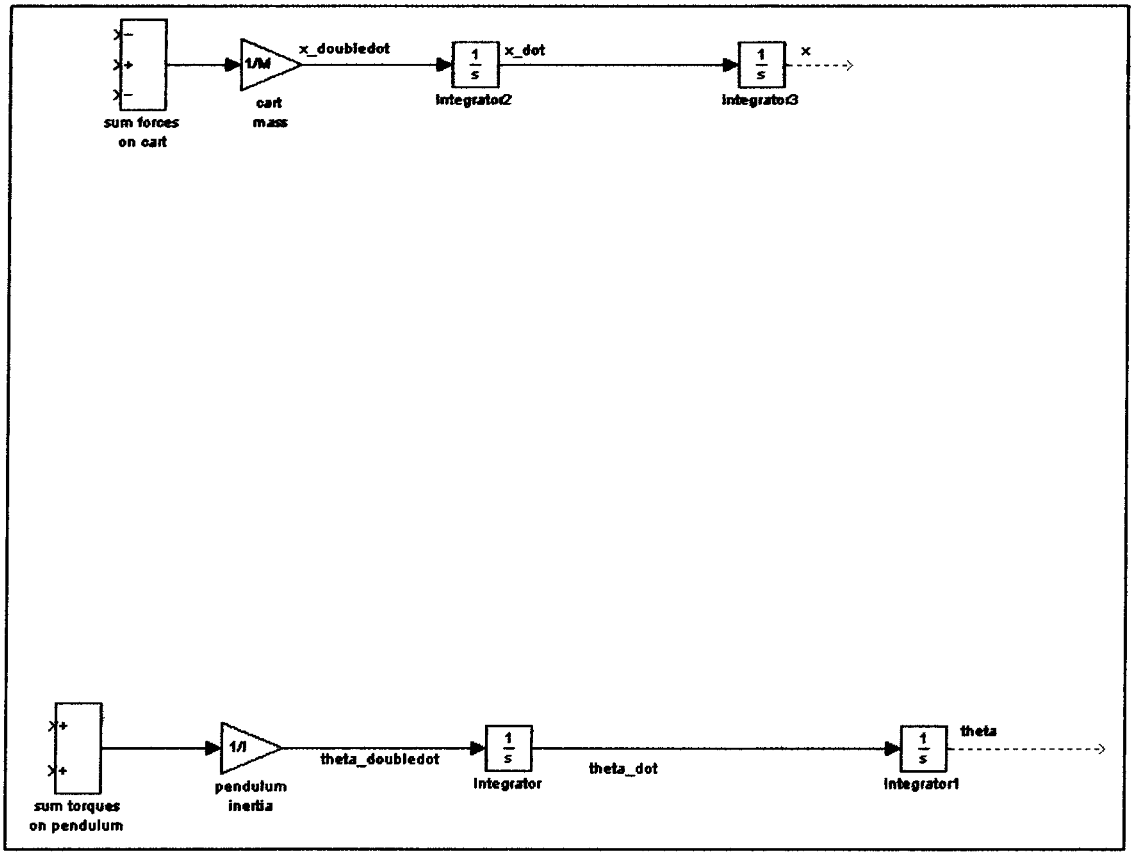

Figure 3.3: Modeling in Simulink

After that, the two forces acting on the cart will be added in the Simulink construction through these steps.

- A Gain block is inserted above the Cart Mass block and its value is changed to "b" and its label to "damping".
- This block can be flipped left-to-right by single clicking on it in order to select it and flip it by selecting Flip Block from the Format menu or from right click or also by hitting Ctrl-F.
- A line is tapped off the "x dot" line and it is connected to the input of the damping block.
- The output of the damping block is connected to the topmost input of the Sum Forces block. The damping force then has a negative sign.