“ I hereby declare that I have read through this report entitle “Transient Analysis of Electrical Circuits using Runge-Kutta Method” and found that it has comply the partial fulfillment for awarding the degree of Bachelor of Mechatronics Engineering”

Signature : ....

Supervisor’s Name : ....

Date : ....

TRANSIENT ANALYSIS OF ELECTRICAL CIRCUITS USING RUNGE-KUTTA METHOD

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A report submitted in partial fulfillment of the requirements for the degree of Bachelor of Mechatronic Engineering(Hons.)

Faculty of Electrical Engineering
UNIVERSITI TEKNIKAL MALAYSIA MELAKA

2017

DECLARATION

I declare that this report entitle "Transient Analysis of Electrical Circuits using Runge-Kutta Method" is the result of my own research except as cited in the references. This report has not been accepted for any degree and is not concurrently submitted in candidature of any other degree.

Signature : ....

Name : ....

Date : ....

To my beloved father and mother

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ABSTRACT

Transient analysis is an analysis of transient response of an electrical circuit. The purpose of transient analysis is to analyse the performance of an electrical circuit. Transient is a sudden application of source to a circuit or a brief increase in current or voltage in a circuit which cause damage to some sensitive components and instruments. Electrical transient phenomena can be generated due to some natural events and switching operations. In order to solve the transient analysis of electric circuit, numerical techniques for utilization in the companion analytical method of transient circuit analysis are used to solve second-order differential equations which generated from circuit equations of RLC circuit. The objective of this project is to solve the transient analysis of electric circuit using analytical method and Runge-Kutta method. Fourth-order Runge-Kutta method is chosen as the numerical method to solve transient analysis of electric circuit due to its high accuracy of approximations. In this project, transient analysis of RLC circuit is simulated using MATLAB and visualised in a graphical form. Next, the results of transient analysis obtained from analytical method and Runge-Kutta method are tabulated and analysed by using Microsoft Excel. Besides that, the aim of this project is to compare the transient analysis obtained form Runge-Kutta method with the analytical solution. There are two numerical simulations conducted in this project. First numerical simulation is to solve transient analysis of electric circuit on three different types of response which are underdamped response, critically damped response and overdamped response. Another numerical simulation is to solve transient analysis of a RLC circuit using different step sizes. The results obtained are analysed and compared in term of error in order to determine the accuracy of Fourth-order Runge-Kutta method. In the first simulation, Fourth-order Runge-Kutta method has a high degree of accuracy which is up to 99.99%. Besides, the accuracy of Fourth-order Runge-Kutta method can be improved as the time step size decimates. Fourth-order Runge-Kutta method provides an alternative solution for transient analysis of electrical circuits.

ABSTRAK

Transient analysis merupakan analisis *transient response* untuk litar elektrik. Tujuan *transient analysis* adalah untuk menganalisis prestasi sesebuah litar elektrik. Transient merupakan aplikasi sumber kepada litar elektrik secara tiba-tiba atau peningkatan yang singkat pada arus atau voltan dalam litar yang menyebabkan kerosakan kepada beberapa komponen dan instrument yang sensitive. Fenomena elektrik transient dapat dihasilkan daripada kejadian alam dan operasi bertukaran. Dalam usaha untuk menyelesaikan *transient analysis* bagi litar elektrik, teknik berangka digunakan bersama dengan kaedah analisis dalam analisis litar elektrik untuk menyelesaikan persamaan pembezaan yang dihasilkan daripada persamaan litar. Tujuan projek ini adalah untuk menyelesaikan *transient analysis* sesebuah litar elektrik dengan menggunakan kaedah analisis dan kaedah Runge-Kutta. Kaedah *Fourth-order Runge-Kutta* dipilih sebagai kaedah berangka yang digunakan untuk menyelesaikan *transient analysis* sesebuah litar elektrik kerana kaedah ini mempunyai tahap ketepatan yang tinggi. Dalam projek ini, MATLAB digunakan untuk mensimulasikan dan menggambarkan *transient analysis* bagi *RLC circuit* dalam bentuk grafik. Seterusnya, Microsoft Excel digunakan untuk merekodkan dan menganalisis keputusan *transient analysis* yang diperolehi daripada kaedah analisis dan kaedah Runge-Kutta. Di samping itu, projek ini bertujuan untuk membandingkan *transient analysis* diperolehi daripada kaedah Runge-Kutta dengan penyelesaian analitikal. Terdapat dua simulasi berangka yang dijalankan dalam projek ini. Simulasi berangka pertama adalah untuk menyelesaikan *transient analysis* litar elektrik ke atas tiga jenis tindak balas yang merupakan *underdamped response, critically damped response* dan *overdamped response.* Simulasi berangka kedua adalah untuk menyelesaikan *transient analysis of RLC circuit* menggunakan saiz langkah yang berbeza. Keputusan yang diperolehi daripada kaedah-kaedah tersebut akan dianalisis dan dibandingkan dari segi ralat untuk menentukan ketepatan kaedah *Fourth-order Runge-Kutta.* Dalam simulasi pertama, kaedah *Fourth-order Runge-Kutta* mempunyai tahap ketepatan yang tinggi iaitu sehingga 99.99%. Selain itu, tahap ketepatan kaedah *Fourth-order Runge-Kutta* dapat diperbaiki dengan menggunakan saiz lengkah masa yang lebih kecil. Kaedah *Fourth-order Runge-Kutta* menyediakan penyelesaian alternative untuk *transient analysis* litar elektrik.

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CHAPTER 1

INTRODUCTION

1.1 General Overview

Electric circuit is known as the path for electric current transmission. Electric circuit is also defined as the interconnection of electric elements or electrical devices [1]. Electrical elements are the electrical components of the circuit. The electrical elements in electric circuit are basically divided into two types which are active elements and passive elements. Generators, batteries and operational amplifiers are the active elements meanwhile resistors, capacitors and inductors are the passive elements. The main difference between the active element and passive element is active element is capable to generate energy while passive element is not.

First-order circuit is characterised by a first-order differential equation. The electric circuit consists of only one single storage element which is a capacitor or an inductor. There are two types of simple circuits which are a circuit comprising a resistor and a capacitor and a circuit comprising a resistor and an inductor. These two types of circuits are well-known as RC circuit and RL circuit. Second-order circuit is characterised by a second-order differential equation. It consists of resistors and the equivalent of two storage elements which are capacitor and inductor. RLC circuit is a typical example of second-order circuits which the three types of passive elements are present in the circuit. All the first-order and second-order circuit are connected in two ways which are in series or in parallel.

In this research, transient analysis of RLC circuit is conducted using Runge-Kutta method. There are multiple applications for RLC circuit and some of the most important applications are oscillator and turners of radio or audio receiver. Besides, RLC circuit is also used to create low-pass filter, high-pass filter, band-stop filter and band-pass filter. Transient analysis is important for an electrical circuit because it is commonly used to analyse the performance of the circuit.

1.2 Motivation

Transient analysis is an analysis of transient response of an electrical circuit. Transient analysis is one of the significant analyses in fundamental electric circuit. The purpose of transient analysis is to analyse the performance of an electrical circuit. Transient analysis of electrical circuit is commonly solved by using analytical method. Analytical solution is defined as the exact solution and usually derived using analysis. Analytical solutions in transient analysis are derived by analysing the circuit using circuit equations and a transient response complete solution is formed.

In this research, numerical methods are proposed to solve the transient analysis of electrical circuit in an alternative way. Numerical solution is an approximate solution obtained by using methods of numerical analysis. In the process of analysing RLC circuit which is a second-order circuit, second-order differential equations is utilized to obtain the transient analysis of the circuit. Runge-Kutta method is chosen as the numerical solution for transient analysis in order to solve the second-order differential equations. Runge-Kutta method is the analogue of Simpson's rule for differential equation. This method is commonly used for numerical analysis due to its high accuracy of approximations. Runge-Kutta method is easy to implement, stable and accurate. However, Runge-Kutta method requires enormous computation time and the error estimation are hard to be done.

Equation (1.1) shows a simple differential equation with the initial condition $x(0) = 1$. The computed point is obtained by simulating equation (1.1) using Fourth-order Runge-Kutta formula with different step size. Table 1.1 shows the error obtained from the equation (1.1) by applying the Fourth-order Runge-Kutta method. Figure 1.1 indicates the graph plotted using fourth-order Runge-Kutta method in application of equation (1.1) with step size of 0.2.

$$\frac{dx}{dt} = x \qquad (1.1)$$

Table 1.1: Error obtained with the Fourth-order Runge-Kutta method [2].

Time Step	Exact Point	Computed Point	Error (%)
$\Delta t = 0.5$	7.389056	7.383970	6.8828×10^{-4}
$\Delta t = 0.2$	7.389056	7.388889	2.2581×10^{-5}
$\Delta t = 0.1$	7.389056	7.389045	1.5335×10^{-6}
$\Delta t = 0.05$	7.389056	7.389055	9.9918×10^{-8}
$\Delta t = 0.02$	7.389056	7.389056	2.6226×10^{-9}
$\Delta t = 0.01$	7.389056	7.389056	1.6528×10^{-10}
$\Delta t = 0.001$	7.389056	7.389056	1.5385×10^{-14}
$\Delta t = 0.0001$	7.389056	7.389056	5.1686×10^{-15}

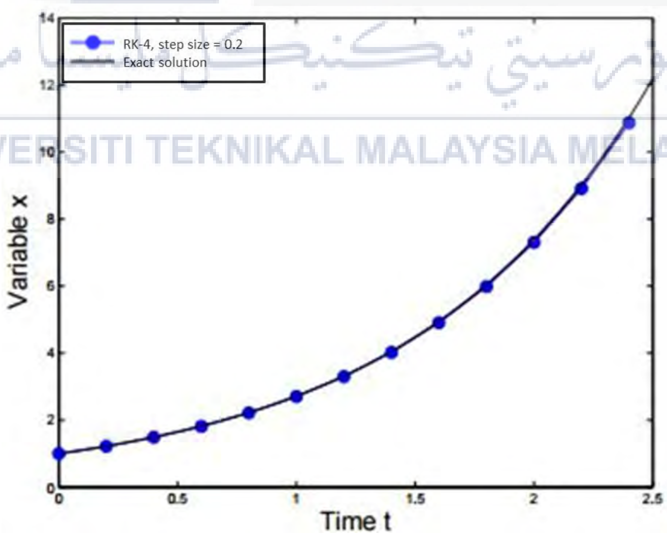

Figure 1.1: Fourth-order Runge-Kutta method in application with step size of 0.2 [2].

According to Table 1.1, as the step size (time) get smaller, the computed point get closer to the exact point and thus reducing the error. Small value of error obtained from the

Fourth-order Runge-Kutta method proves that this numerical method has a high degree of accuracy. In conclusion, Fourth-order Runge-Kutta method is suitable for solving differential equation because of its high degree of accuracy even with complex and high order differential equation.

1.3 Problem Statement

The complete response of an electric circuit can be decomposed into two components which are transient response and steady-state response [1]. Transient response is the temporary response of a circuit that will recede with time. Transient is a sudden application of source to a circuit or a brief increase in current or voltage in a circuit which cause damage to some sensitive components and instruments. Electrical transient phenomena can be generated due to some natural events and switching operations. One of the examples of the natural events is lightning strikes or surges and the switching operations mentioned are capacitor, load and transformer energizing [3]. The transient response can be categorized into three types of damping which are underdamped, critically damped and overdamped.

In previous research, transient analysis of electrical circuits is done in few methods which are the analytical method, Cauchy-Heaviside (C-H) operational method, Fourier Transformation method and Laplace Transformation method. C-H operational method is not as systematic as Laplace Transformation method. Hence, it has been abandoned in favour of the Laplace Transformation method. Laplace Transformation method is commonly used to solve differential equation especially in cases with discontinuous forcing terms or a periodic, non-sinusoidal forcing term [4]. Analytical method which worked with transient analysis of electric circuit requires higher knowledge of mathematics and less physical knowledge or matter related with transient behaviour of electric circuit compared to the others [5].

In order to solve the transient analysis of electric circuit, numerical techniques for utilization in the companion analytical method of transient circuit analysis are used to solve the differential equation. Numerical method is usually used to solve the differential equation of a second-order circuit. Numerical method is able to overcome the limitation of

Laplace Transformation method which numerical method is able to solve more complicated equation such as higher order differential equation. Besides, graph can be plotted easily based on the function given by using this method.

1.4 Objectives

The main objectives of conducting the project are:-

1. To solve the transient analysis of electric circuit using analytical solution and Runge-Kutta method.
2. To compare the transient analysis obtained from Runge-Kutta method with the analytical solution.
3. To validate the transient analysis of the electrical circuits obtained from Runge-Kutta method.

1.5 Scope of Project

The scopes of the research are listed as follow:-

1. Transient analysis conducted is emphasized only on second-order series RLC circuit.
2. Three types of transient analysis are conducted which are underdamped response, critically damped response and overdamped response.
3. Transient analysis is solved using analytical solution which derived from circuit analysis and equations.
4. The results of transient analysis are tabulated and analysed by using MATLAB software.
5. The results of transient analysis are performed in the graph of voltage against time.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Generally, the project named Transient Analysis of Electrical Circuit using Runge-Kutta method utilizes many resources from previous works of researchers. In this chapter, there are three major parts which are the transient analysis, analytical solution and types of numerical methods.

2.2 Background

2.2.1 Transient Analysis

Transients are the currents and voltages vary with time resulting from an unusual application of source. Transient is usually occurred due to switching [6]. There are many different analysis might be applied to the analogue circuit simulation and one of the examples is transient analysis. Transient analysis is the analysis most commonly used to analyse circuit performance. Besides, transient response analysis of doubly-fed induction generator (DFIG) is conducted during three-phase symmetrical grid voltage swell. Transient operation and steady-state operation are the two processes included in the transient process of DFIG. A series of transient processes is caused by the stator voltage swell and the transient processes are referring to overvoltage of DC-link, rotor impulse current and torque oscillation [7].

Transient response for non-uniform transmission line is usually analysed by using the theory of uniform transmission line. Transient analysis of transmission line becomes more significant as the connection line of integrated circuit. The main reason is transient analysis is applied to ensure the normal operation of high speed electronic information system [8].

A new Lightning Activity Monitoring System (LAMS) is installed on the mountain Lovćen. The new monitoring system is subjected to standard lightning impulse test to determine its reliability and energy capability when there is a lightning surge. Transient analysis due to the lightning event is applied on the wind turbine blade [9]. The transient caused by the lightning event can be very destructive for the system based on the result of simulations obtained.

In conclusion, many applications demand the result of transient response for systems. There are many applications of transient analysis such as double-fed induction generator (DFIG), non-uniform transmission line and Lightning Activity Monitoring System (LAMS). The importance of transient analysis is to determine the performance and stability of a system.

2.2.2 General Concepts of Electric Circuits

In transient analysis of electrical circuits, some fundamental laws that govern electric circuits are used to generate circuit equations for second-order circuits. There are some basic circuit equations frequently used for solving the electric circuit such as Ohm's law, Kirchhoff's Voltage Law (KVL) and Kirchhoff's Current Law (KCL) [1].

Ohm's law is defined as the voltage, v across a resistor is directly proportional to the current, i passes through the resistor. The mathematical form of Ohm's law is given by

$$v = iR \tag{2.1}$$

where v is voltage across resistor, i is current passes through resistor and R is resistance of resistor.

Since Ohm's law is not sufficient to analyse an electric circuit by itself, it is combined with Kirchhoff's law to analyse a large variety of electric circuits. Kirchhoff's first law is based on the law of conservation of charge while Kirchhoff's second law is based on the principle of conservation.

Kirchhoff's first law which is known as Kirchhoff's Current Law (KCL) states that the algebraic sum of currents entering and leaving a node is zero. In other words, the sum of current entering a node is equal to the sum of currents leaving the node. Kirchhoff's Current Law can be used to obtain the currents for each branch when the electric circuit elements are connected in parallel. Mathematically, Kirchhoff's first law is defined as

$$\sum_{n=1}^{N} i_n = 0 \qquad (2.2)$$

where N is the number of branches connected to the node and i_n is the nth current in or out of the node.

Kirchhoff's second law which is known as Kirchhoff's Voltage Law (KVL) is shown as equation (2.3) where M is the number of voltages in the loop or closed path and v_m is the mth voltage. This law states that the algebraic sum of voltages across various circuit elements in a closed path or loop is zero. In other words, the sum of voltage drops is equal to the sum of voltage rises. This law can be applied to obtain the total voltage of the circuit when voltage sources are connected in series.

$$\sum_{m=1}^{M} v_m = 0 \qquad (2.3)$$

2.2.2.1 Analysing of Second-Order Circuits

In second-order circuits, the initial conditions for circuit variables and their derivatives are important to analyse the circuits. The RLC circuits are considered for series circuits and parallel circuits in analysing transient response of circuits.

Equation (2.4) shows the capacitor voltage for the initial condition where $t = 0^-$ indicates the time just before a switching event and $t = 0^+$ indicates the time just after the switching event. The switching event is assumed taking place at $t = 0$. According to the equation below, the capacitor voltage is always continuous. This equation is used to find the initial condition of a RLC circuit especially for those variables that cannot change abruptly.

$$v(0^+) = v(0^-) \tag{2.4}$$

Equation (2.5) indicates the inductor current for the initial condition where $t = 0^-$ is the time just before a switching event and $t = 0^+$ denotes the time just after the switching event. The switching event is assumed taking place at $t = 0$. Based on equation (2.5), the inductor current is always continuous. This equation is used to find the initial condition of a RLC circuit.

$$i(0^+) = i(0^-) \tag{2.5}$$

A series RLC circuit with a voltage source is shown in Figure 2.1. Capacitor is a passive element that does not dissipate energy but store energy in its electric field meanwhile inductor is a passive element that does not dissipate energy but store energy in its magnetic field.

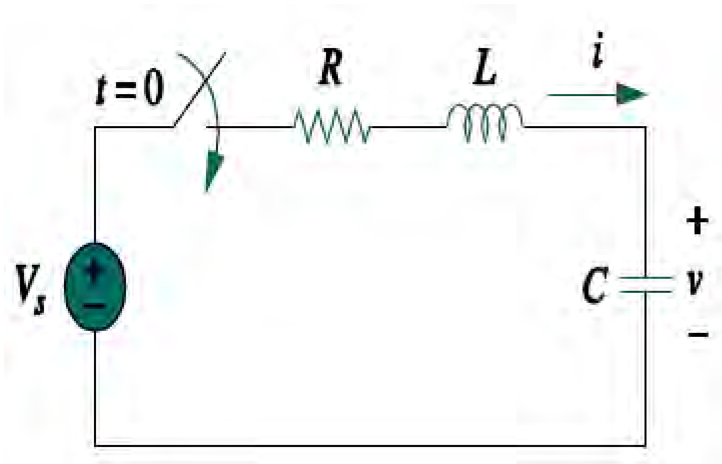

Figure 2.1: Series RLC Circuit

In order to solve the RLC circuit, the current-voltage relationship for a capacitor is obtained. Equation (2.6) and (2.7) shows the current-voltage relationship for a capacitor where i is the current supplied by a capacitor, C is the capacitance of the capacitor, v is the voltage across the capacitor and t is a period of time.

$$i = C\frac{dv}{dt} \tag{2.6}$$

or

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt \tag{2.7}$$

In order to obtain the circuit equation for series RLC circuit as shown in the Figure 2.1, Kirchhoff's Voltage Law (KVL) is applied around the loop for $t > 0$. The circuit equations of series RLC circuit as shown in Figure 2.1 are as following:

$$V_s = V_R + V_L + V_C \tag{2.8}$$

and hence

$$V_s = Ri + L\frac{di}{dt} + v \tag{2.9}$$

where V_s is the voltage source of RLC circuit, V_R, V_L and V_C are the voltage across resistor, inductor and capacitor respectively, R is the resistance of resistor, L is the inductance of inductor, v is the voltage and $\frac{di}{dt}$ is the derivative of current passes through inductor with respect to time. The equation (2.8) and (2.9) are formed to generate a second-order differential equation for the circuit.

The second-order differential equation of the RLC circuit as shown the Figure 2.1 are shown as equation (2.10) where v is the voltage, R is the resistance of resistor, L is inductance of inductor, C is the capacitance of capacitor and V_s is the voltage source of RLC circuit. Equation (2.10) is formed by substituting equation (2.6) into equation (2.9)

$$\frac{d^2v}{dt^2} + \frac{Rdv}{Ldt} + \frac{v}{LC} = \frac{V_s}{LC} \tag{2.10}$$

2.2.3 Analytical solutions of second-order ODE

In previous research, there are few methods utilised to solve second-order ordinary differential equations (ODEs) which are Laplace Transformation method, undetermined coefficients and variation of parameters. Laplace Transformation method is commonly used to solve differential equation especially in cases with discontinuous forcing terms or a periodic, non-sinusoidal forcing term. Laplace transformation method offers a systematic and routine approach for differential equation. However, there are some limitations of Laplace transform. The application of inverse Laplace transform is complicated. The complicated equation is difficult to be solved by using Laplace transform. Laplace transform is not capable to interpret spectra in a clear manner directly from the equation given [4].

The method of undetermined coefficients is a straightforward solution for higher-order linear differential equations which only involving techniques of college algebra and differential. Undetermined coefficients method is used to solve ordinary differential equation due to its simple implementation [10]. This method is able to reduce the problem to an algebra problem which is easier to be solved. However, this method is only useful for constant coefficient differential equations and it only work for a fairly small class of functions [11].

Variation of parameters is a general solution for non-homogeneous linear ordinary differential equations. This method can be used in any case but it is necessary to obtain complementary solution to solve the cases. Besides, a couple of integrals need to be done in order to complete the method. The problems are unable to be solved if the complementary solution is not obtained or the integrals are too complex [12].

2.2.4 Numerical Methods

2.2.4.1 Euler Method

Particle dynamics simulation is used to simulate the behaviour of different kinds of particle physics phenomena such as thermal, shocks, solid or fluid interaction, water, gas and others. Numerical integration is vital in the simulation to solve the differential equation of moving objects. The performance of the simulation will be affected by the integration method used. In this paper, the computational results of different integration method are discussed and compared. Euler method is used in most of the particle dynamics simulation studies. Euler method is the simplest integration method. This method has a low degree of accuracy due to two kinds of error which are truncation error and cumulative error. Besides, this method is only used for smooth particle moving but the particles usually have an extreme abrupt movement. Therefore, other integration methods might be used to overcome this problem. The computational time of Euler method is faster than other methods like Heun method and Fourth-order Runge-Kutta method [13].

In order to solve initial value problem in ordinary differential equations, L.Y. Xu and etc. proposed a new method and the new method is compared with the classical numerical methods such as Euler method. Based on the results of calculation, Euler method has a low level of precision [14].

2.2.4.2 Heun's Method

Molecular dynamics simulation is used to determine the motion of a molecule into the models of three different phases which are gases, liquids and solids. By using this simulation, the position, orientation and velocities change of a molecule with respect to time are described. Heun's method is applied on the mathematical modelling of diatomic molecule O2 to solve the model and study the motion behaviour of the molecule. The numerical stability of Heun's method will be affected by the condition of stability of the model. Heun's method is unstable for the condition of stability which is unsatisfied [15].

According to the case study by J.G. Tan and etc., analytical solution of a dynamical system which driven by Poisson white noise (PWN) is unfrequently obtained. Hence, a numerical method is developed for the dynamical system in order to find response samples. There are three steps in the numerical methods and one of the steps is calculating the system output according to the generated input samples using Heun's method [16].

The same numerical method is proposed by W.X. Lin and etc. as a solver for state-space models in the dq frame and abc frame and compared to other solvers such as Dommel's method and Runge-Kutta method. The accuracy of Heun's method is lost if the simulation step is increased from 2μs to 4μs. Besides, Heun's method has lower level of accuracy compared to Dommel's method and this method is not recommended for AVM modelling. By comparing the three numerical methods, Heun's method is the least accurate method [17].

Differential equations play significant role in describing and solving a situation with ordinary differential. Numerical methods are commonly applied for solving stochastic differential equations. In order to solve stochastic differential equations, numerical stability and convergence of the numerical methods are considered [18]. Heun's method is applied to perform the numerical integration of the stochastic differential equations. Heun's method is reduced to the standard second-order Runge-Kutta method for the pure deterministic case [19]. Based on the numerical studies for simple spin systems, Heun's method is numerically more stable and allows greater time step size than the Euler method [20, 21].

The same numerical solution is proposed by Y.F Liu and etc. to analyse the power system dynamics under random excitation. Heun's method and Euler-Maruyama (EM) method are applied to the linearised SDE model and compared. Heun's solution is better than Euler-Maruyama method in fitting the analytical solution of a stochastic differential equation. Besides, Heun's method able to obtain accurate simulation results and has a higher numerical stability compared to Euler-Maruyama method. However, Euler-Maruyama method requires less computation than Heun's method which may reduce the calculation errors [22].

2.2.4.3 Runge-Kutta Method

Runge-Kutta method is a useful numerical method to solve ordinary differential equations (ODE). In this paper, the equation of transient flow of an ideal gas through porous media which is one of the examples of non-linear partial differential equation (PDE) is considered. The results obtained from Runge-Kutta method are compared with the analytical solution obtained by implicit, explicit and Crank-Nicholson (C-N) finite difference methods. Runge-Kutta method has higher degree of accuracy and yields better results in comparison to other methods. Besides, the error committed by Runge-Kutta method is less than other methods. However, the execution time and core storage requirements in this method are more than the C-N finite difference method [23].

Differential equations are frequently utilised to model problem in science and engineering. L.Y. Xu and etc. proposed a new method for solving initial value problems in ordinary differential equations (ODEs) and compared the new method with the classical numerical methods such as Euler method and Runge-Kutta method. Fourth-order Runge-Kutta method has its shortcomings where it requires a large amount of computation as it has four evaluations per step. However, the results of calculation using Runge-Kutta method are more precise compared to the new method [14].

Runge-Kutta method is suggested by X.X Jiang and etc. and applied in the implementation of the Extended Kalman Filter on FPGA. Runge-Kutta method is used to approximate the continuous time update and solve the Riccati equation. This numerical method is able to provide accurate results. However, it is expensive to implement due to more computation are required by Runge-Kutta method [24].

Runge-Kutta method is applied in the time-domain simulation of non-linear circuits to replace the conventional multistep integration method. This method is suitable to be implemented in an electrical simulator. Due to the numerical instability of the conventional multistep integration method, the results obtained can be inaccurate and unreliable. In order to overcome the limitations, Runge-Kutta method is considered to improve and enrich the numerical engine of time-domain simulators. The RK method is able to achieve high accuracy order while maintaining asymptotically stability property [25].

Runge-Kutta method is proposed in [17] as a solver for state-space models in the “dq” frame and “abc” frame and compared to other solvers such as Dommel’s method and Heun’s method. The accuracy of Runge-Kutta method is higher than Dommel’s method and Runge-Kutta method is able to give accurate responses with almost 100 times of improvement in the simulation speed.

2.2.4.4 Comparison over Numerical Methods

Table 2.1: Comparison over Numerical Methods.

Methods	Euler Method	Heun's Method	Runge-Kutta Method
Application	• Particle dynamics simulation [13]	• Molecular dynamics simulation [15] • Computation of power system dynamics [22]	• Transient flow of ideal gas [23] • Time-domain simulation of non-linear circuits [25]
Advantages	• Simplest integration method • Fast computational time	• High numerical stability • Accurate simulation results	• High accuracy • Stable • Reliable
Disadvantages	• Only for smooth particle moving • Low degree of accuracy	• More computation than EM method • Less precise compared to RK method	• Expensive to implement • Higher computational time

Euler method is commonly used in particle dynamics simulation. It is the simplest integration method among the three methods. It has fast computational simulation but low degree of accuracy. Heun's method is applied on molecular dynamics simulation and computation of power system dynamics. This method is numerically more stable than Euler method and able to obtain accurate simulation results. However, the results obtained from Heun's method are less precise than Runge-Kutta method. Runge-Kutta method is utilised in many applications although it has higher computational time.

In conclusion, Runge-Kutta method is chosen as the numerical method for solving the transient analysis of electrical circuit. Runge-Kutta method has a high accuracy of approximation. Besides, this method is stable and the results obtained are reliable.

CHAPTER 3

METHODOLOGY

3.1 Introduction

In Chapter 3, the techniques and equations utilized in the project are discussed. Firstly, methodology section covers the flowchart of the final year project. Gantt chart is also presented the project's timeline as planned within the first and second semester of research. The analysing methods of the project are discussed in this chapter to generate suitable equations for analysing transient response of the electric circuit using the method proposed.

3.2 Project Flowchart

The project flowchart consists of flowchart of the overall project. The overall project methodology flowchart is constructed as shown in the Figure 3.1.

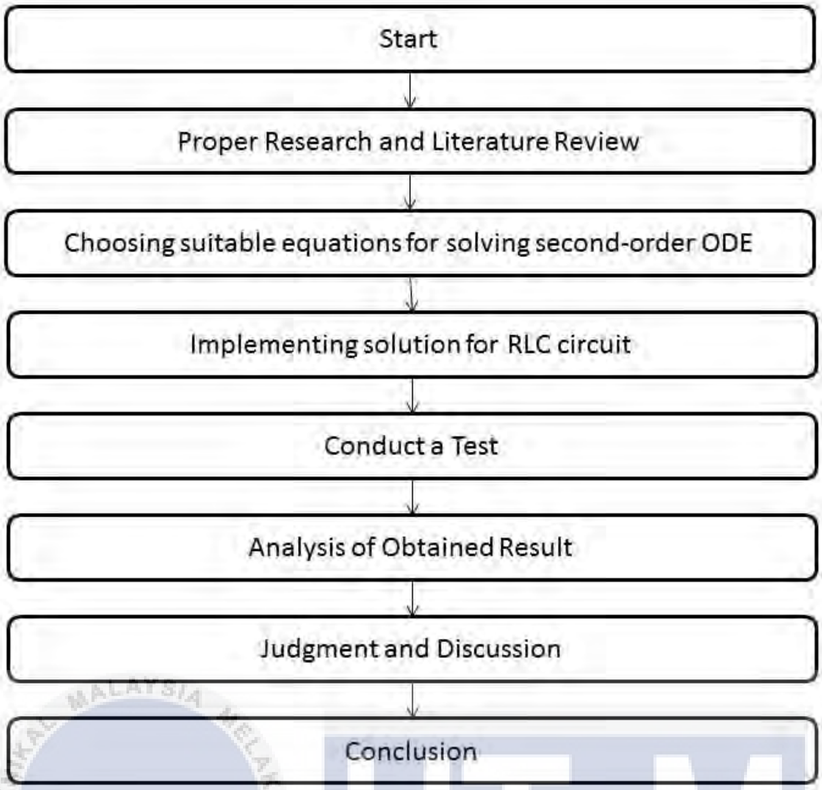

Figure 3.1: Project Flowchart

3.3 Gantt Chart

Project Gantt chart describes the time flow of the overall project and the delegated task within the specific amount period of time. Both of the Gantt Chart for FYP 1 and FYP 2 is attached in APPENDIX D.

3.4 Analysing Methods

In this part, the analysing methods can be divided into two major parts which are analytical solution and numerical solution for transient analysis of electrical circuits. Analytical solution is the exact solution derived by analysing the circuit using circuit equations. Numerical solution is the approximation solution used to solve the transient analysis with second-order differential equations that generated from circuit equations of RLC circuit.

3.4.1 Analytical Solution

Analytical solution of transient analysis is generated from circuit analysis and equations. The analytical solution is used to solve the transient analysis of second-order RLC circuit for three different types of transient response which are underdamped response, critically damped response and overdamped response.

Equation (3.1) is the characteristic equation obtained from equation (2.10) by replacing the first derivative by s and the second derivative by s^2. In this equation, R is the resistance of resistor, L is inductance of inductor and C is the capacitance of capacitor. The equation is used to obtain the roots of the characteristic equation and consider three cases for the transient response [1].

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{3.1}$$

The characteristic equation as shown in equation (3.1) is a quadratic equation. In order to solve the quadratic equation, a quadratic formula is used. The quadratic formula is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.2}$$

where a is the coefficient of second derivative, b is the coefficient of first derivative and c is the coefficient of constant.

The roots of the characteristic equation using the quadratic formula as shown in the equation (3.2) are defined as

$$s_i = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \text{for } i = 1,2 \tag{3.3}$$

where s_i is known as natural frequencies because they are associated with the natural response of the circuit, R is the resistance of resistor, L is inductance of inductor and C is the capacitance of capacitor.

The roots of the characteristic equations which in term of α and ω_0 are defined as equation (3.4) and the α and ω_0 are shown in the equation (3.5) where α is the neper frequency or damping factor and ω_0 is the resonant frequency or strictly as the undamped natural frequency.

$$s_i = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} \qquad \text{for } i = 1,2 \tag{3.4}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{3.5}$$

Based on equation (3.4), there are three types of cases for the transient response obtained from the solution which are overdamped case, critically damped case and underdamped case.

From equation (3.3) and (3.4), when $\alpha > \omega_0$, $C > 4L/R^2$, both roots of the characteristic equation are negative and real. Hence, the transient response of the circuit is an overdamped case. The equation of the transient response is given by

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{3.6}$$

and hence,

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{3.7}$$

Equation (3.6) is the transient response for overdamped case where $v_t(t)$ is the transient response of overdamped case, s_1 and s_2 are the roots of the characteristic equation (3.3), A_1 and A_2 are the constants to be determined and t is the time. Equation (3.7) is the complete solution for overdamped case where V_s is source voltage of the circuit, $v(t)$ is the total response of overdamped case, s_1 and s_2 are the roots of the characteristic equation (3.3), A_1 and A_2 are the constants to be determined and t is the time.

From equation (3.3) and (3.4), when $\alpha = \omega_0$, $C = 4L/R^2$, the roots of the characteristic equation are real and equal. Therefore, the transient response of the circuit is a critically damped case. The equation of the transient response is shown as following:

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{3.8}$$

and hence,

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \tag{3.9}$$

Equation (3.8) is the transient response for critically damped case where $v_t(t)$ is the transient response of critically damped case, A_1 and A_2 are the constants to be determined, α is the damping factor and t is the time. Equation (3.9) is the complete solution for critically damped case where V_s is source voltage of the circuit, $v(t)$ is the total response of critically damped case, A_1 and A_2 are the constants to be determined, α is the damping factor and t is the time.

From equation (3.3) and (3.4), when $\alpha < \omega_0$, $C < 4L/R^2$, the transient response of the RLC circuit is an underdamped case. The roots of the characteristic equation are complex and expressed as equation as shown in equation (3.10).

$$s_i = -\alpha \pm j\omega_d \qquad \text{for } i = 1,2 \tag{3.10}$$

where

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} \tag{3.11}$$

The response of the circuit is shown in equation (3.12). Equation (3.12) is the transient response for underdamped case where $v_t(t)$ is the transient response of underdamped case, A_1 and A_2 are the constants to be determined, α is the damping factor, ω_d is the damping frequency and t is the time. Equation (3.13) is the complete solution for underdamped case where V_s is source voltage of the circuit, $v(t)$ is the total response of underdamped case, A_1 and A_2 are the constants to be determined, α is the damping factor, ω_d is the damping frequency and t is the time.

$$v_t(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \tag{3.12}$$

and hence,

$$v(t) = V_s + e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \tag{3.13}$$

Damping ratio is defined as the ratio of damping factor and resonant frequency. Damping ratio is defined as

$$\xi = \frac{\alpha}{\omega_0} \tag{3.14}$$

where ξ is damping ratio, α is damping factor and ω_0 is resonant frequency. This equation is a useful parameter to determine the type of transient response of the RLC circuit. The relationship between damping ratio and type of transient response is shown in Table 3.1.

Table 3.1: Relationship between Damping Ratio and Types of Transient Response

Damping Ratio	**Types of Transient Response**
$\xi < 1$	Underdamped
$\xi = 1$	Critically damped
$\xi > 1$	Overdamped

3.4.2 Numerical Methods for Ordinary Differential Equations

Numerical method is a mathematical approach designed as the solution for numerical problems. Numerical methods are commonly used to solve complex problems by providing an alternative way. Numerical methods for ordinary differential equations are techniques used to find numerical approximations to solve ordinary differential equations. In this research, numerical methods for ordinary differential equations are utilised to solve the second-order differential equations that generated from the RLC circuit which shown as equation (2.10). The numerical methods used are Euler method, Heun's method and Fourth-order Runge-Kutta method.

3.4.2.1 Formulation of Euler Method

Euler method is a numerical method that generates a table of approximate value of the function that solves the initial value problem. This method is the most straightforward method for solving differential equations. The Euler method for second-order differential equation is written as

$$y_{i+1} = y_i + hf(x_i,\ y_i,\ z_i) \tag{3.15}$$

$$z_{i+1} = z_i + hg(x_i,\ y_i,\ z_i) \tag{3.16}$$

where $i = 0, 1, 2 \ldots n$, h is the step size, $f(x_i, y_i, z_i)$ is the function of f and $g(x_i, y_i, z_i)$ is the function of g [26, 27].

3.4.2.2 Formulation of Heun's Method

The adaptive Heun's method equation for solving second-order ordinary differential equation is shown as following:

$$y_{i+1} = y_i + \frac{1}{2}h(k_1 + k_2) \tag{3.17}$$

$$z_{i+1} = z_i + \frac{1}{2}h(l_1 + l_2) \tag{3.18}$$

where

$$k_1 = f(x_i,\ y_i,\ z_i) \tag{3.19}$$

$$l_1 = g(x_i,\ y_i,\ z_i) \tag{3.20}$$

$$k_2 = f(x_i + h,\ y_i + k_1 h,\ z_i + l_1 h) \tag{3.21}$$

$$l_2 = g(x_i + h,\ y_i + k_1 h,\ z_i + l_1 h) \tag{3.22}$$

Equation (3.17) and (3.18) are the formula of Heun's method for second-order ordinary differential equations where $i = 0, 1, 2 \dots n$. The equations from (3.19) to (3.22) are substituted into the equation (3.17) and (3.18) where h is the step size and k_m and l_m are the increment based on the slope at the beginning and end of the interval [26, 27].

3.4.2.3 Formulation of Fourth-order Runge-Kutta Method

Runge-Kutta method is one of the methods of solving ordinary differential equations. The fourth-order Runge-Kutta method is used to solve the second-order circuits. The adaptive Runge-Kutta method equation for solving second-order ordinary differential equation is shown as following:

$$y_{i+1} = y_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.23}$$

$$z_{i+1} = z_i + \frac{1}{6}h(l_1 + 2l_2 + 2l_3 + l_4) \tag{3.24}$$

where

$$k_1 = f(x_i,\ y_i,\ z_i) \tag{3.25}$$

$$l_1 = g(x_i,\ y_i,\ z_i) \tag{3.26}$$

$$k_2 = f(x_i + \frac{h}{2},\ y_i + \frac{k_1 h}{2},\ z_i + \frac{l_1 h}{2}) \tag{3.27}$$

$$l_2 = g(x_i + \frac{h}{2},\ y_i + \frac{k_1 h}{2},\ z_i + \frac{l_1 h}{2}) \tag{3.28}$$

$$k_3 = f(x_i + \frac{h}{2},\ y_i + \frac{k_2 h}{2},\ z_i + \frac{l_2 h}{2}) \tag{3.29}$$

$$l_3 = g(x_i + \frac{h}{2},\ y_i + \frac{k_2 h}{2},\ z_i + \frac{l_2 h}{2}) \tag{3.30}$$

$$k_4 = f(x_i + h,\ y_i + k_3 h,\ z_i + l_3 h) \tag{3.31}$$

$$l_4 = g(x_i + h,\ y_i + k_3 h,\ z_i + l_3 h) \tag{3.32}$$

Equation (3.23) and (3.24) are the formula of Fourth-order Runge-Kutta method for second-order differential equations where $i = 0, 1, 2 \dots n$. The equations from (3.25) to (3.32) are substituted into the equation (3.23) and (3.24) where h is the step size and k_m and l_m are the increment based on the slope at the beginning, midpoint and end of the interval [6, 26, 27].

3.5 Experiments

3.5.1 Experiment Setup for Series RLC Circuit

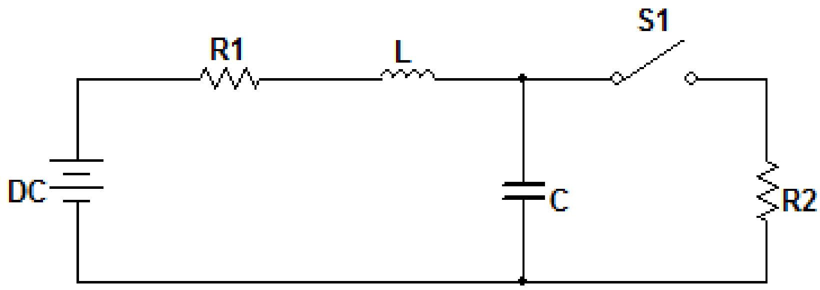

Figure 3.2: Series RLC circuit

In this experiment, a series RLC circuit is set up as shown in Figure 3.2. The experiment is carried out with three different conditions as shown below. The main purpose of this experiment is to present three types of graphs which are underdamping, critically damping and overdamping. The parameters for the RLC circuit such as resistance, inductance and capacitance are set follow the condition 1, 2 and 3.

Condition 1: $\xi < 1$

Condition 1 is also known as underdamped response. For this condition, the series RLC circuit must fulfil the condition $\xi = \frac{\alpha}{\omega_0} < 1$ or $\alpha < \omega_0$ where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$. The parameters as shown in the Figure 3.2 is adjusted to fulfil the desired condition.

Condition 2: $\xi = 1$

The condition 2 is actually a critically damped response. The parameters of series RLC circuit as shown in the Figure 3.2 is adjusted to fulfil the condition of $\xi = \frac{\alpha}{\omega_0} = 1$ or $\alpha = \omega_0$ where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$.

Condition 3: $\xi > 1$

Overdamped response is set as the condition 3 for this experiment. To fulfil the condition, damping ratio, ξ must be greater than 1 or α must be greater than ω_0 where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$. The values of the resistance, capacitance and inductance are taken into account to meet the targeted condition.

3.5.2 Numerical Simulation of Transient Analysis of Electrical Circuit

3.5.2.1 Numerical Simulation 1: Comparison between Analytical Solution and Numerical Methods

In the experiment, a series RLC circuit is set up as shown in Figure 3.2. Different types of transient analysis of RLC circuit would be conducted to study the nature of numerical methods on different conditions. The conditions mentioned are referred to underdamped response, critically damped response and overdamped response. The main purpose of this experiment is to conduct numerical simulations of transient analysis using three different numerical methods and determine the accuracy of each numerical method. The numerical methods proposed are Euler method, Heun's method and Fourth-order Runge-Kutta method.

Firstly, some calculations are done in order to find out the suitable values of the parameters for the three conditions. Next, the transient responses of each condition are solved by using the equations from equation (2.1) to (2.10) and (3.1) to (3.14). The solutions of transient response are shown in APPENDIX A. The complete solutions for each condition are considered as the equations for the analytical solution.

MATLAB is utilised to design and create code for each numerical simulation. The numerical simulation is conducted for the entire numerical methods and a graph of transient analysis is generated as shown in Figure 3.5.

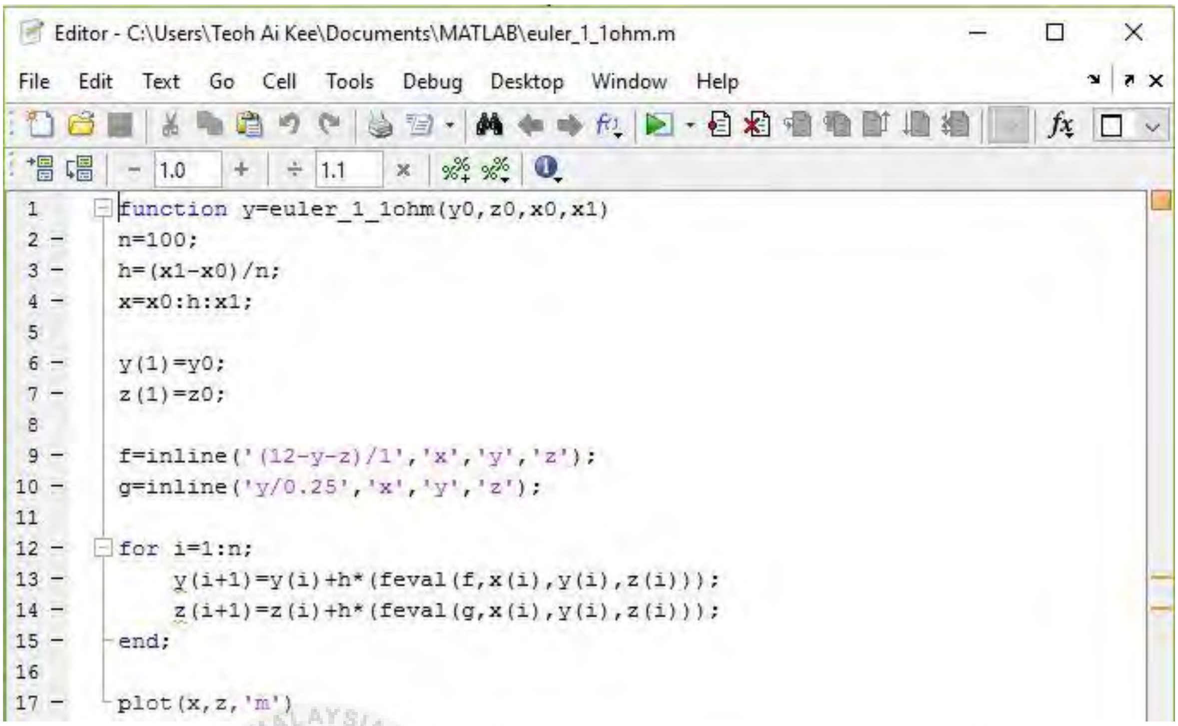

Figure 3.3: MATLAB IDE for Euler method simulation.

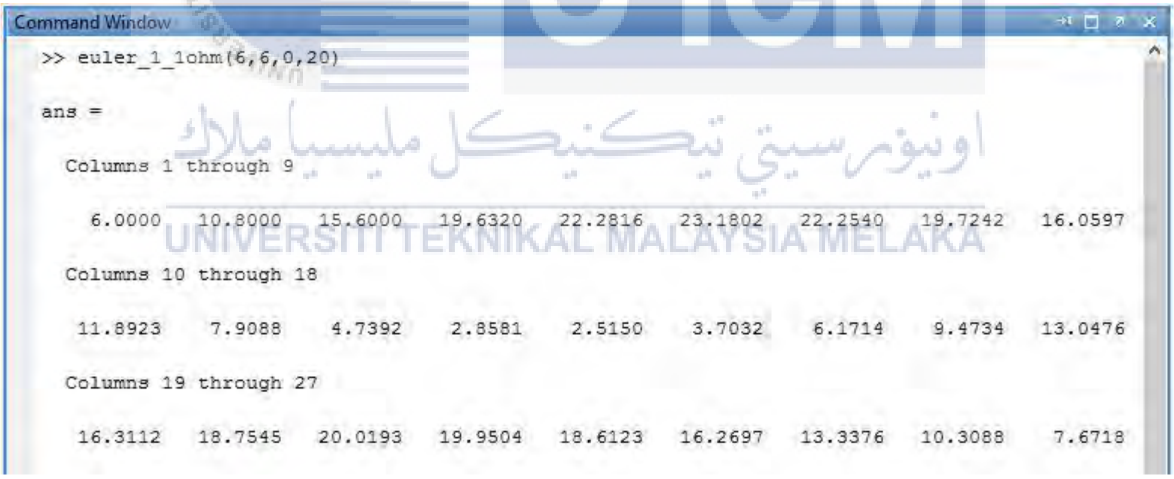

Figure 3.4: Command Window in MATLAB.

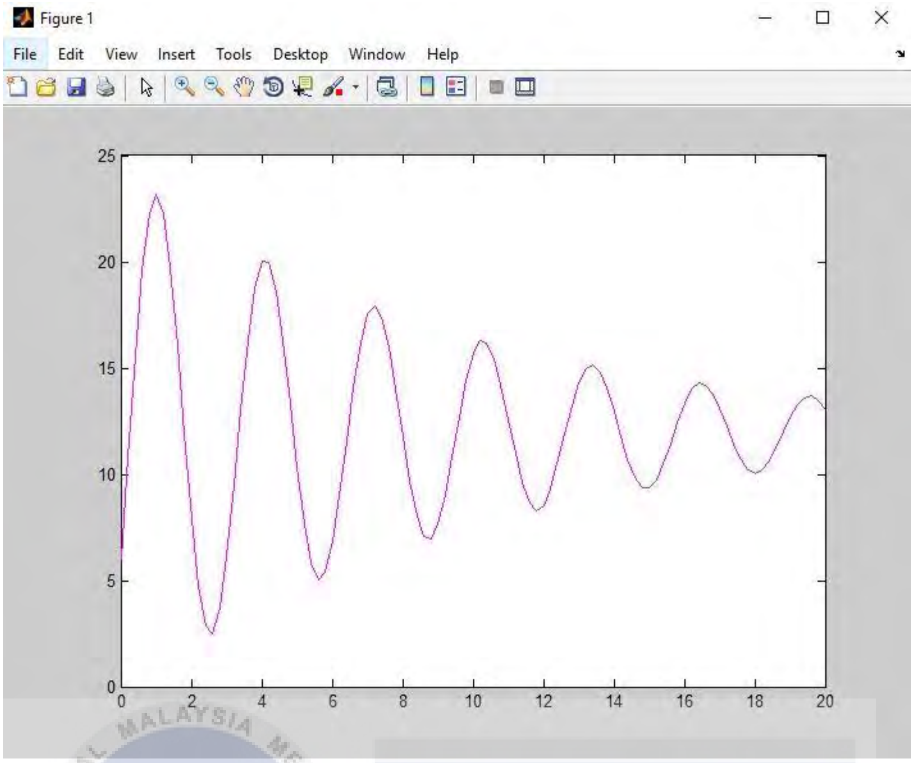

Figure 3.5: Graph of Voltage against Time using Euler method.

All the waveforms generated are displayed in a same graph to compare the accuracy of the numerical simulations against the analytical solution. All the raw data from each simulation are extracted and analysed based on its error value (APPENDIX C).

Error is expressed as absolute error or as percentage of error. Absolute error is defined as the difference between the actual value of the variable and the measured value of the variable [28]. Mathematically, absolute error may be defined as

$$e = |Y_n - X_n| \tag{3.33}$$

where e is absolute error, Y_n is the actual value of the variable and X_n is the measured value of the variable. Therefore, percentage of error may be defined as

$$\%\, error = \left|\frac{Y_n - X_n}{Y_n}\right| \times 100\% \tag{3.34}$$

where Y_n is the actual value of the variable and X_n is the measured value of the variable. In this research, the actual value of the variable is referred to computed point of analytical solution meanwhile the measure value of the variable is referred to the computed point of Euler method, Heun's method and Fourth-order Runge-Kutta method.

3.5.2.2 Numerical Simulation 2: Comparison of Fourth-order Runge-Kutta method with different step size, *h*

In the experiment, a series RLC circuit is set up as shown in Figure 3.2. The main purpose of this experiment is to conduct numerical simulations of transient analysis using Fourth-order Runge-Kutta method with different step sizes and determine the accuracy of the method based on the step size.

Firstly, the transient analysis of electric circuit is solved by using the equations from equation (2.1) to (2.10) and (3.1) to (3.14). The equation of transient response is considered as the equation for the analytical solution.

MATLAB is utilised to design and create code for Fourth-order Runge-Kutta method. The numerical simulation is conducted for the entire numerical method with different step size and a graph of transient analysis is generated as shown in Figure 3.8.

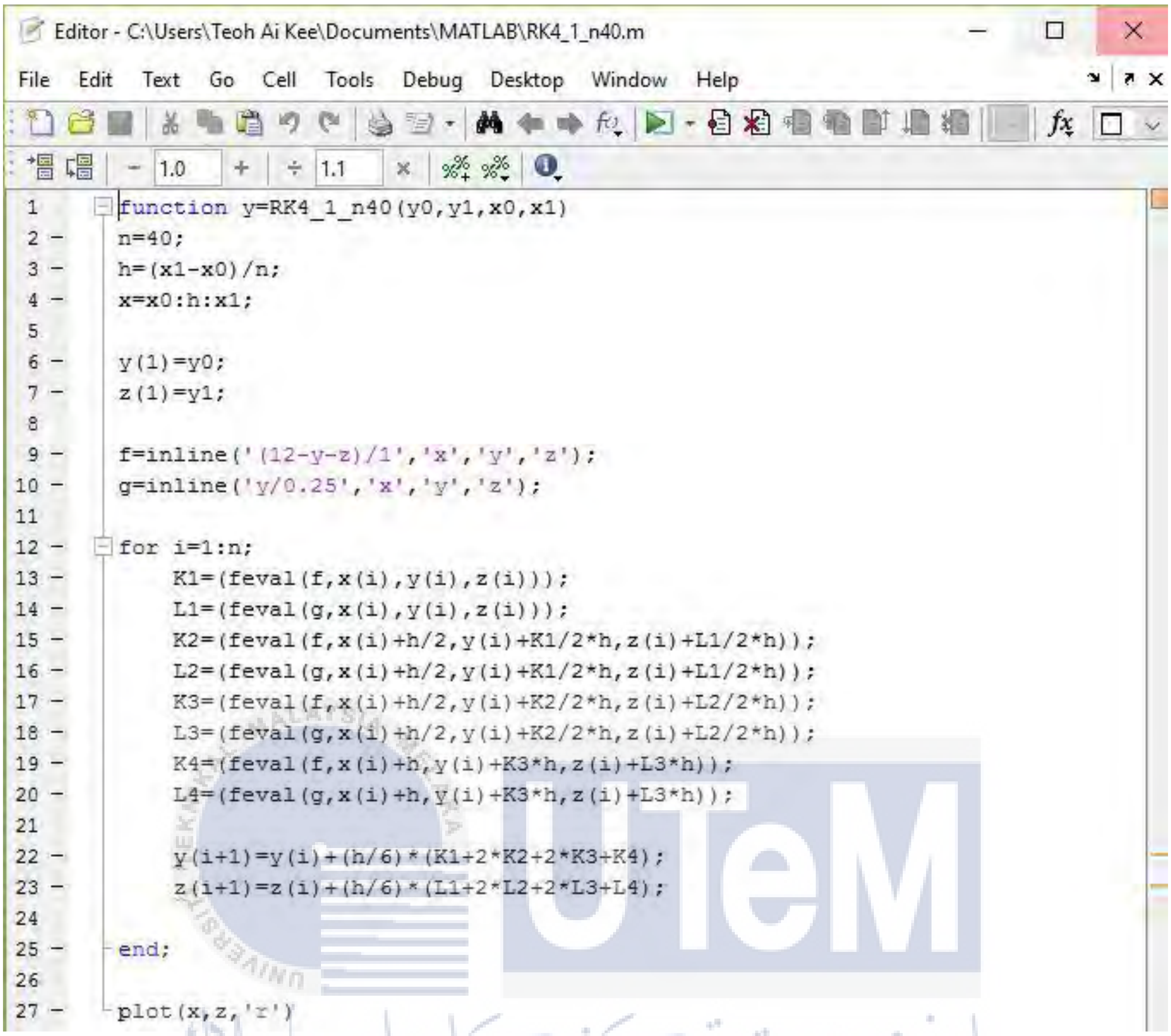

Figure 3.6: MATLAB IDE for RK-4 method simulation.

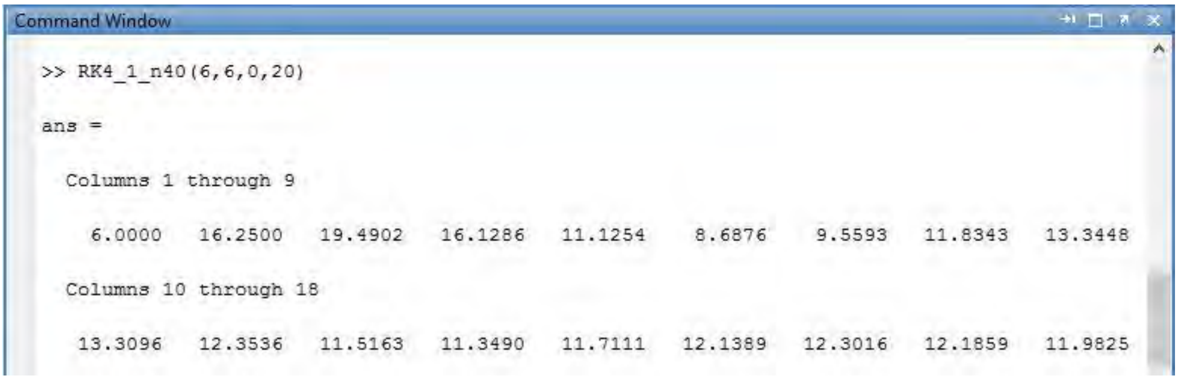

Figure 3.7: Command Window in MATLAB.

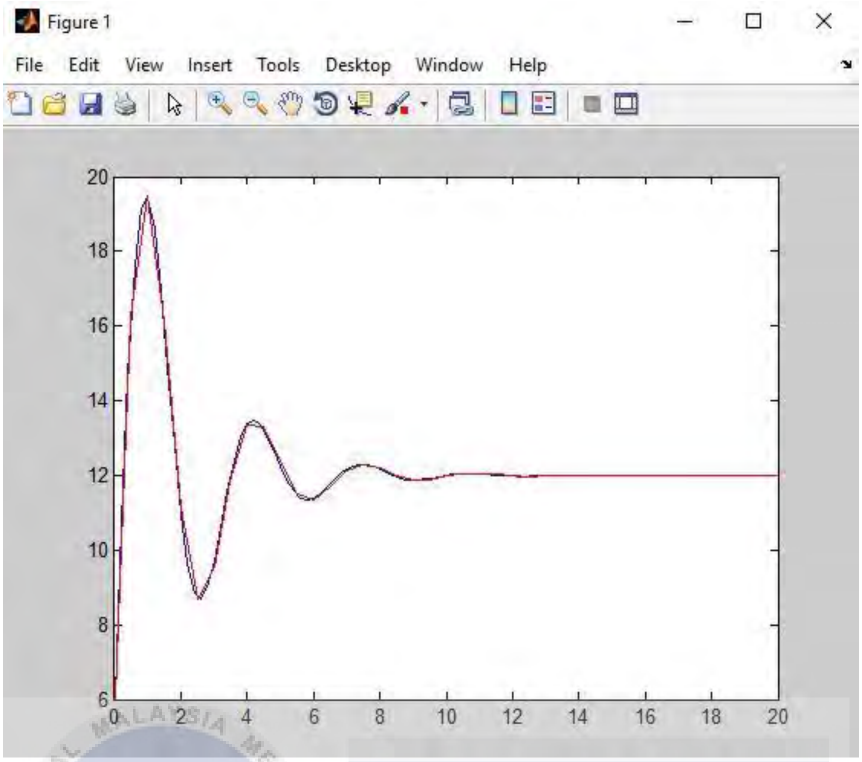

Figure 3.8: Graph of Voltage against Time using analytical method and RK-4 method.

All the waveforms generated are displayed in a same graph to compare the accuracy of the Fourth-order Runge-Kutta method with certain step size against the analytical solution. All the raw data from each simulation are extracted and analysed based on its error value (APPENDIX C).

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, the discussion will focus on the outcome retrieved from the numerical simulations done. The numerical simulation results are tabulated in a table form and compared with the analytical solution. Furthermore, analysis and general deduction is also made from the result.

4.2 Numerical Simulation of Transient Analysis of Electrical Circuit

4.2.1 Comparison between Analytical Solution and Numerical Methods

In the experiment, a series RLC circuit is set up as shown in Figure 3.2. In order to obtain different types of response, the experiment is carried out with three different conditions. In each condition, the parameter of resistor $R1$ varies meanwhile the parameters of capacitor, C and inductor, L are fixed. Some calculations are done in order to find out the suitable values of the parameters for the three conditions and the values of electrical elements are shown in Table 4.1.

Table 4.1: The values of the electrical elements of series RLC circuit for three conditions.

Element	Value		
	Condition 1	**Condition 2**	**Condition 3**
Type of Response	Underdamped	Critically damped	Overdamped
DC voltage source	12 V	12 V	12 V
Resistor, *R1*	< 4 Ω	4 Ω	> 4 Ω
Resistor, *R2*	1 Ω	1 Ω	1 Ω
Inductor, *L*	1 H	1 H	1 H
Capacitor, *C*	0.25 F	0.25 F	0.25 F

Since the accuracy of the numerical methods will be affected by the time step size, the time step size will be constant for all the numerical methods used. In this experiment, the simulation time for all the conditions are from 0s to 20s with the step size of 0.2s. In order to compare the accuracy of the numerical methods, the computed points of the three numerical methods are taken at the specific time. The computed point of each method is compared with the analytic solution. Besides that, the maximum error, minimum error and average error are taken into account as another method to determine the accuracy of numerical methods.

4.2.1.1 Condition 1: Underdamped Response

In condition 1, the resistance, *R1* must be less than 4Ω in order to achieve underdamped response. In this part, the experiment is repeated three times by considering 1Ω, 2Ω and 3Ω as the value of resistor, *R1* for the circuit. In order to obtain the best numerical method for transient analysis of RLC circuit, three numerical methods are compared with the analytical solution. The three numerical methods mentioned are Euler method, Heun's method and Fourth-order Runge-Kutta method.

An initial point is significant for numerical methods in solving second-order different equations. In this experiment, the initial point is referred to the initial voltage and initial current of RLC circuit. The initial voltage and initial current are calculated by using the Ohm's law stated as equation (2.1) and the calculations are done as shown in APPENDIX A.

Table 4.2: Initial voltage and initial current for the RLC circuit with resistance of 1Ω, 2Ω and 3Ω.

Initial condition	**Resistance, R1**		
	1Ω	**2Ω**	**3Ω**
Initial voltage, $v(0)$	6 V	4 V	3 V
Initial current, $i(0)$	6 A	4 A	3 A

Figure 4.1, Figure 4.2 and Figure 4.3 show the transient analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method with different value of resistance, *R1*. The minimum error, maximum error and average error between analytical solutions and each numerical method are calculated based on the raw data and shown in Table 4.3, Table 4.4 and Table 4.5.

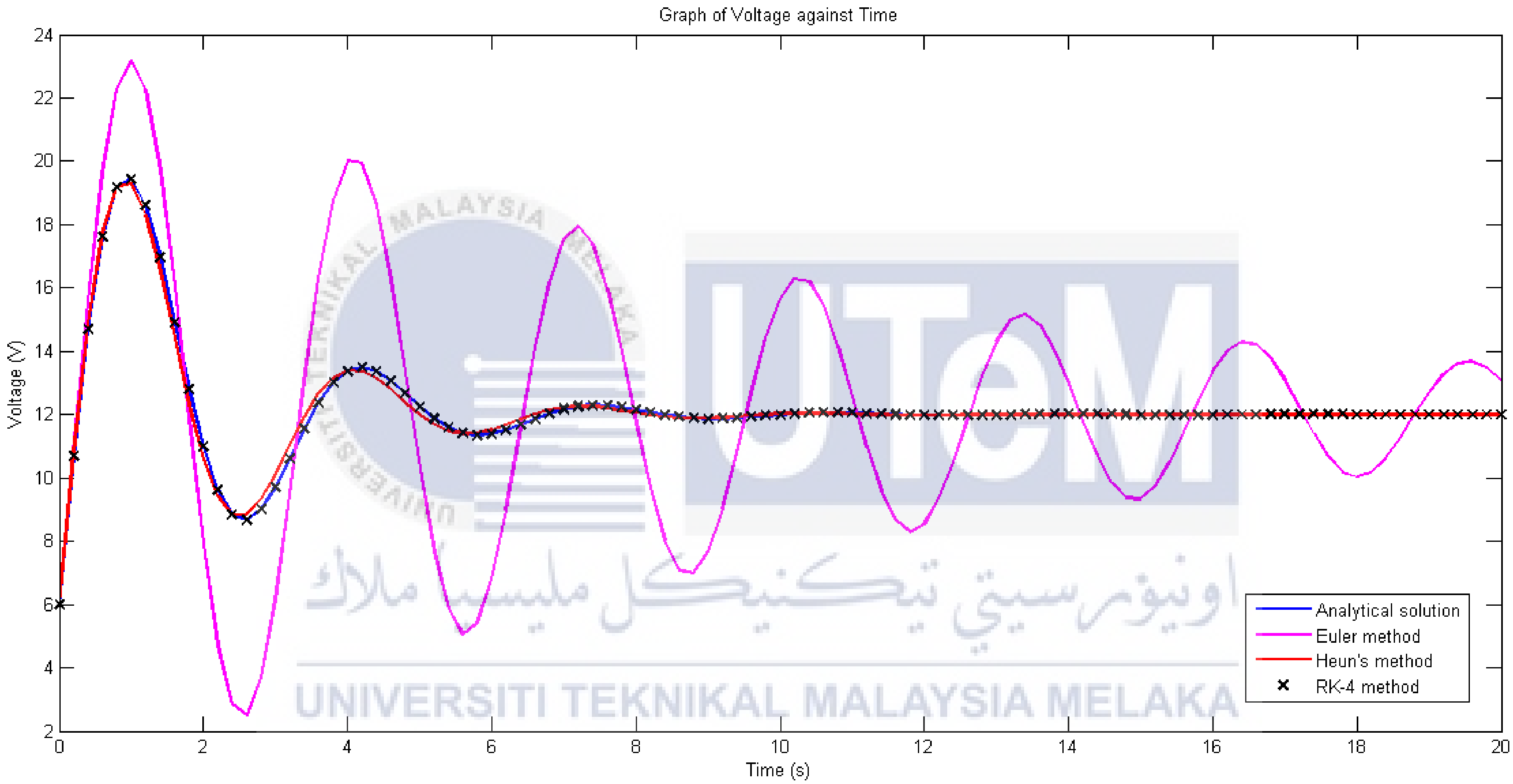

Figure 4.1: Transient Analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method ($R1$=1Ω, L=1H, C=0.25F).

Table 4.3: The minimum error, maximum error and average error between analytical solution and numerical methods ($R1$=1Ω, L=1H, C=0.25F).

Error	Numerical Methods		
	Euler method	**Heun's method**	**RK-4 method**
Minimum	0.00%	0.00%	0.00%
Maximum	71.02%	4.27%	0.04%
Average	21.80%	0.69%	0.01%

Figure 4.1 shows the transient analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method where the resistance, $R1$ is 1Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and each numerical method are calculated and shown in Table 4.3.

Based on Table 4.3, the minimum errors for all three numerical techniques are 0.00% which indicated that there is no difference between the computed points of analytical solution and all the three numerical techniques. The minimum errors for all numerical methods are 0.00% because of the initial point of the entire line graphs are on the same point which is 6V. The maximum error of transient analysis using Heun's method is 4.27% and the average error is 0.69%. By using Fourth-order Runge-Kutta method in solving transient analysis of electric circuit, the maximum error is 0.04% and the average error is 0.01%. As seen from Figure 4.1, Euler method has the highest maximum error and average error which are about 71.02% and 21.80% respectively.

A computed point is taken at 1s from each line graph in Figure 4.1 in order to compare the accuracy of the numerical simulations on transient analysis. From Figure 4.1, the voltage of RLC circuit gained from the analytical solution at 1s is 19.4439V whereas the voltage gained from the Euler method, Heun's method and Fourth-order Runge-Kutta method are 23.1802V, 19.2988V and 19.4464V respectively. According to the computed points taken, the percentage of error in transient analysis using Euler method is 19.22% which is the highest among the other numerical methods. Heun's method has a percentage of error of 0.75% meanwhile RK-4 method has a percentage of error of 0.01%.

According to Figure 4.1, it is clearly shown that when a computed point is taken at 1s from each line graph, Euler method has the worst accuracy of numerical simulation on the transient analysis of RLC circuit which is about 80.78%. The accuracy of the numerical simulation can be improved by using other numerical methods which are Heun’s method and Fourth-order Runge-Kutta method. By using Heun’s method, the error between analytical solution and numerical method is reduced about 18.47%. The error between analytical solution and numerical method is reduced the most when using Fourth-order Runge-Kutta method which the reduction of error is about 19.21%.

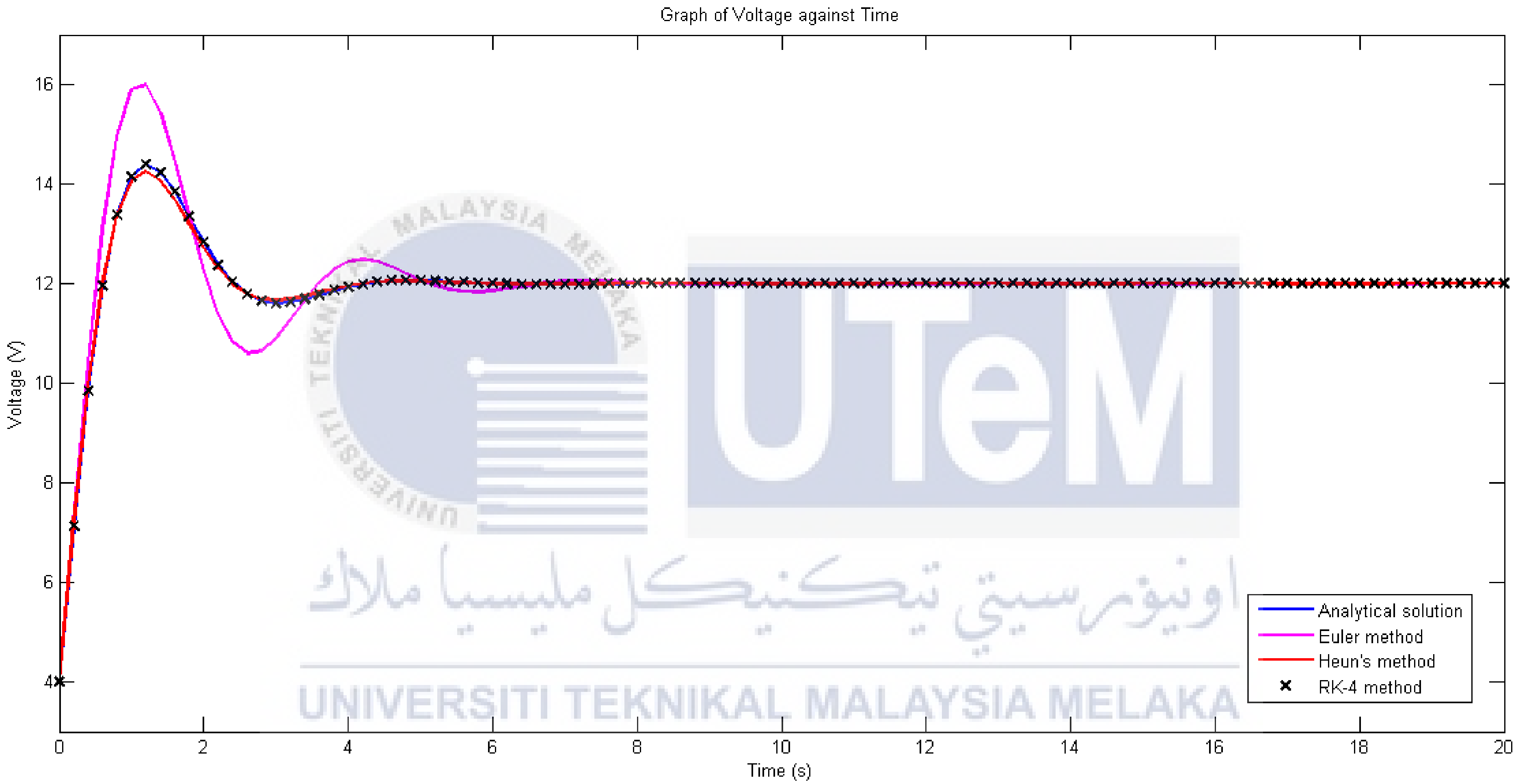

Figure 4.2: Transient Analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method (*R1*=2Ω, *L*=1H, *C*=0.25F)

Table 4.4: The minimum error, maximum error and average error between analytical solution and numerical methods ($R1$=2Ω, L=1H, C=0.25F).

Error	**Numerical Methods**		
	Euler method	**Heun's method**	**RK-4 method**
Minimum	0.00%	0.00%	0.00%
Maximum	12.40%	1.27%	0.01%
Average	1.47%	0.13%	0.00%

Figure 4.2 shows the transient analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method where the resistance, $R1$ is 2Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and each numerical method are calculated and shown in Table 4.4.

From Table 4.4, it is clearly shown that the minimum errors for all three numerical techniques are 0.00%. The minimum errors for all numerical methods are 0.00% because of the initial point of the entire line graphs are on the same point which is 4V. The maximum error of transient analysis using Euler method is 12.40% and the average error is 1.47%. The maximum error and average error obtained using Euler method are the highest among the three numerical methods. By using Heun's method in generating a graph of transient analysis of electrical circuit, the maximum error is 1.27% and the average error is 0.13%. RK-4 method has the lowest maximum error and average error which are approximately 0.01% and 0.00% respectively.

A computed point is taken at 1s from each line graph in Figure 4.2 in order to compare the accuracy of the numerical simulations on transient analysis. According to the Figure 4.2, the voltage of RLC circuit obtained from the analytical solution at 1s is 14.1497V meanwhile the voltage obtained from the Euler method, Heun's method and Fourth-order Runge-Kutta method are 15.9040V, 14.0643V and 14.1513V respectively. As reported in APPENDIX C, the percentage of error in transient analysis using Euler method is 12.40% which is the highest among the other numerical methods. Heun's method has a percentage of error of 0.60% and RK-4 method has a percentage of error of 0.01%.

From Figure 4.2, it is clearly shown that when a computed point is taken at 1s from each line graph, Euler method has the least accuracy of numerical simulation on the transient analysis of RLC circuit which is about 87.60%. As seen from the figure and table above, the accuracy of the numerical simulation can be improved by using other numerical methods which are Heun's method and Fourth-order Runge-Kutta method. By using Heun's method, the error between analytical solution and numerical method is reduced about 11.80%. The error between analytical solution and numerical method is reduced the most when using Fourth-order Runge-Kutta method which the reduction of error is about 12.39%.

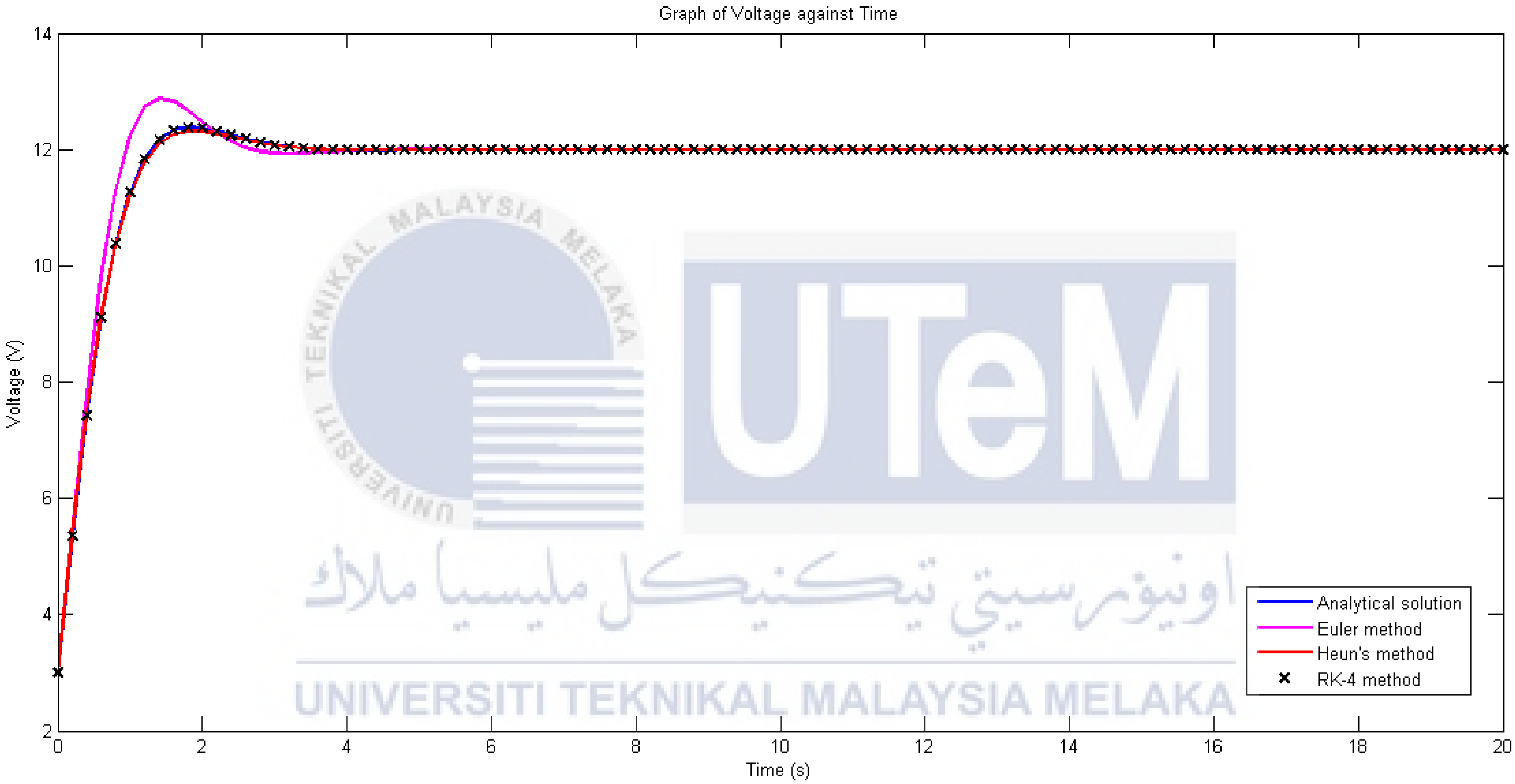

Figure 4.3: Transient Analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method ($R1$=3Ω, L=1H, C=0.25F)

Table 4.5: The minimum error, maximum error and average error between analytical solution and numerical methods ($R1$=3Ω, L=1H, C=0.25F).

Error	**Numerical Methods**		
	Euler method	**Heun's method**	**RK-4 method**
Minimum	0.00%	0.00%	0.00%
Maximum	8.88%	1.03%	0.01%
Average	0.59%	0.07%	0.00%

Figure 4.3 shows the transient analysis of RLC circuit using analytical solution, Euler method, Heun's method and RK-4 method where the resistance, $R1$ is 3Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and each numerical method are calculated and shown in Table 4.5.

Based on Table 4.5, the minimum errors for the entire numerical methods are 0.00%. The minimum errors are about 0.00% for all numerical methods due to same initial point of the entire line graphs which the initial point is 3V. From Table 4.5, it is clearly shown that the lowest maximum error and average error of numerical simulation on transient analysis are obtained by Fourth-order Runge-Kutta method. The maximum error and average error obtained by RK-4 method are about 0.01% and 0.00% correspondingly. Heun's method has the maximum error of 1.03% and the average error of 0.07%. By using Euler method as a numerical simulation on transient analysis of electrical circuit, the maximum error gained is 8.88% and the average error is 0.59% which are the highest among the numerical techniques used.

A computed point is taken at 1s from each line graph in Figure 4.3 in order to compare the accuracy of the numerical simulations on transient analysis. According to the Figure 4.3, the voltage of RLC circuit obtained from the analytical solution at 1s is 11.2620V meanwhile the voltage obtained from the Euler method, Heun's method and Fourth-order Runge-Kutta method are 12.2352V, 11.2127V and 11.2629V respectively. As reported in APPENDIX C, the percentage of error in transient analysis using RK-4 method is the lowest among the other numerical methods which is 0.01%. The percentage of error of Heun's method is about 0.44% at this computed point. Euler method has the highest percentage of error which is approximately 8.64%.

According to Figure 4.3, it is clearly shown that when a computed point is taken at 1s from each line graph, Euler method has the least accuracy of numerical simulation on the transient analysis of RLC circuit which is about 91.36%. As seen from the figure and table above, the accuracy of the numerical simulation can be improved by using other numerical methods which are Heun's method and Fourth-order Runge-Kutta method. By using Heun's method, the accuracy of numerical simulation on the transient analysis of RLC circuit is improved about 8.20%. The accuracy of numerical simulation on the transient analysis of the RLC circuit has the most improvement which is approximately 8.63% when using Fourth-order Runge-Kutta method in transient analysis.

Based on Figure 4.1 to Figure 4.3, as the transient analysis of RLC circuit is approaching critically damped response, the fluctuation of the waveforms of the transient analysis using Euler method decreases. As the oscillation of the waveforms generated by Euler method decreases, the accuracy of Euler method is improved. Euler method is commonly used to solve non-stiff equation. According to Figure 4.1, waveform that generated by Euler method oscillates frequently before reaching steady-state and hence the solution of Euler method is numerically unstable for this case. The maximum error of transient analysis using Euler method is the highest which is about 71.02% compared to Heun's method and RK-4 method. The second-order differential equation of this case is an example of stiff equation where the solution being sought is varying slowly but the analytical solution is varying rapidly. Besides, Euler method has the slow convergence of the series. If the numerical solution of a numerical method approaches the exact solution as the step size, h goes to zero, the numerical method is said to be convergent. In this numerical simulation, the time step size can be decimated in order to improve the accuracy of Euler method.

Based on the results and analysis obtained from the numerical simulations, Fourth-order Runge-Kutta method has the highest degree of accuracy compared to Euler method and Heun's method. Therefore, RK-4 method is chosen as the numerical method to run the numerical simulation of transient analysis for the following conditions.

4.2.1.2 Condition 2: Critically damped Response

In critically damped response, the resistance, *R1* must be equal to 4Ω in order to fulfill the condition. The experiment is conducted by taking 4Ω as the value of resistor, *R1* for the circuit. In this part, results of Fourth-order Runge Kutta method are compared with the analytical solution. The initial voltage and initial current are calculated by using the Ohm's law stated as equation (2.1) and the calculations are done as shown in APPENDIX A. Table 4.6 shows the initial voltage and initial current for the RLC circuit with resistance of 4Ω.

Table 4.6: Initial voltage and initial current for the RLC circuit with resistance of 4Ω.

Initial condition	**Resistance, R1**
	4Ω
Initial voltage, $v(0)$	2.4 V
Initial current, $i(0)$	2.4 A

Figure 4.4 shows the transient analysis of RLC circuit using analytical solution and RK-4 method where the resistance, *R1* is 4Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and RK-4 method are shown in Table 4.7.

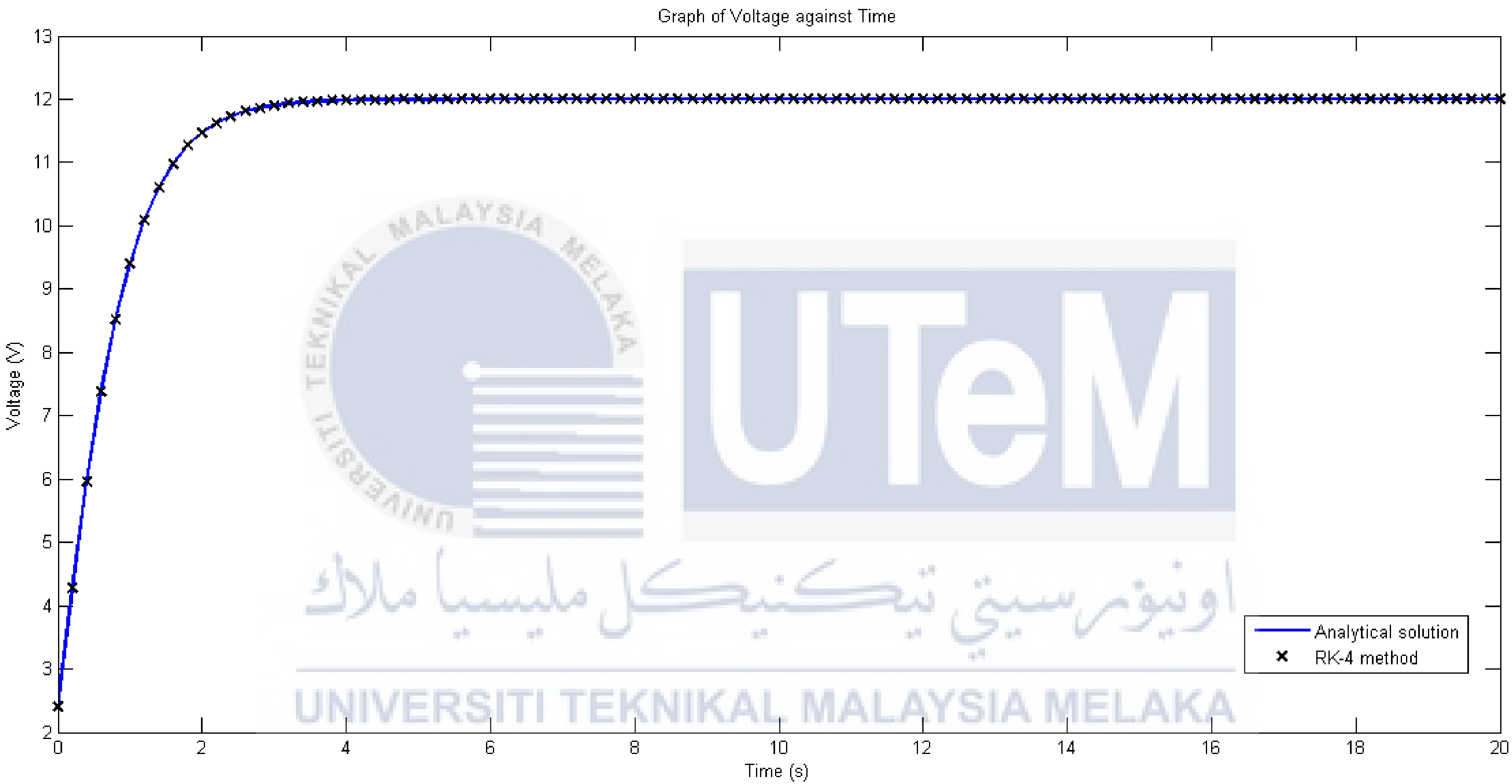

Figure 4.4: Transient Analysis of RLC circuit using analytical solution and RK-4 method (*R1*=4Ω, *L*=1H, *C*=0.25F)

Table 4.7: The minimum error, maximum error and average error between analytical solution and RK-4 method ($R1$=4Ω, L=1H, C=0.25F).

Error	Numerical Method
	RK-4 method
Minimum	0.00%
Maximum	0.03%
Average	0.00%

A computed point is taken at 1s from each line graph in Figure 4.4 in order to determine the accuracy of Fourth-order Runge-Kutta method on transient analysis. According to the Figure 4.4, the voltage of RLC circuit obtained from the analytical solution at 1s is 9.4016V meanwhile the voltage obtained from the Fourth-order Runge-Kutta method is 9.4021V. As reported in results (APPENDIX C), the percentage of error in transient analysis using RK-4 method is 0.01%.

Based on Table 4.7, the minimum error for transient analysis using Fourth-order Runge-Kutta method is 0.00% due to same initial point of the entire line graphs which the initial points of analytical solution and RK-4 method are 2.4V. By using Fourth-order Runge-Kutta as a numerical simulation on transient analysis of electrical circuit, the maximum error obtained is about 0.03% and the average error is about 0.00%. Since the error obtained by RK-4 method is minute, RK-4 method is considered as a high accuracy numerical method which the accuracy is up to 99.97% in this critically damped case.

4.2.1.3 Condition 3: Overdamped Response

In condition 3, the resistance, $R1$ must be more than 4Ω in order to achieve overdamped response. In this part, the experiment is repeated two times by considering 5Ω and 10Ω as the value of resistor, $R1$ for the circuit. In this part, results of Fourth-order Runge Kutta method are compared with the analytical solution. The initial voltage and initial current are calculated by using the Ohm's law stated as equation (2.1) and the calculations are done as shown in APPENDIX A. Table 4.8 shows the initial voltage and initial current for the RLC circuit with resistance of 5Ω and 10Ω.

Table 4.8: Initial voltage and initial current for the RLC circuit with resistance of 5Ω and 10Ω.

Initial condition	**Resistance, R1**	
	5Ω	**10Ω**
Initial voltage, $v(0)$	2 V	1.0909 V
Initial current, $i(0)$	2 A	1.0909 A

Figure 4.5 and Figure 4.6 show the transient analysis of RLC circuit using analytical solution and RK-4 method with different value of resistance, $R1$. The minimum error, maximum error and average error of Fourth-order Runge Kutta method against analytical solution are calculated and shown in Table 4.9 and Table 4.10.

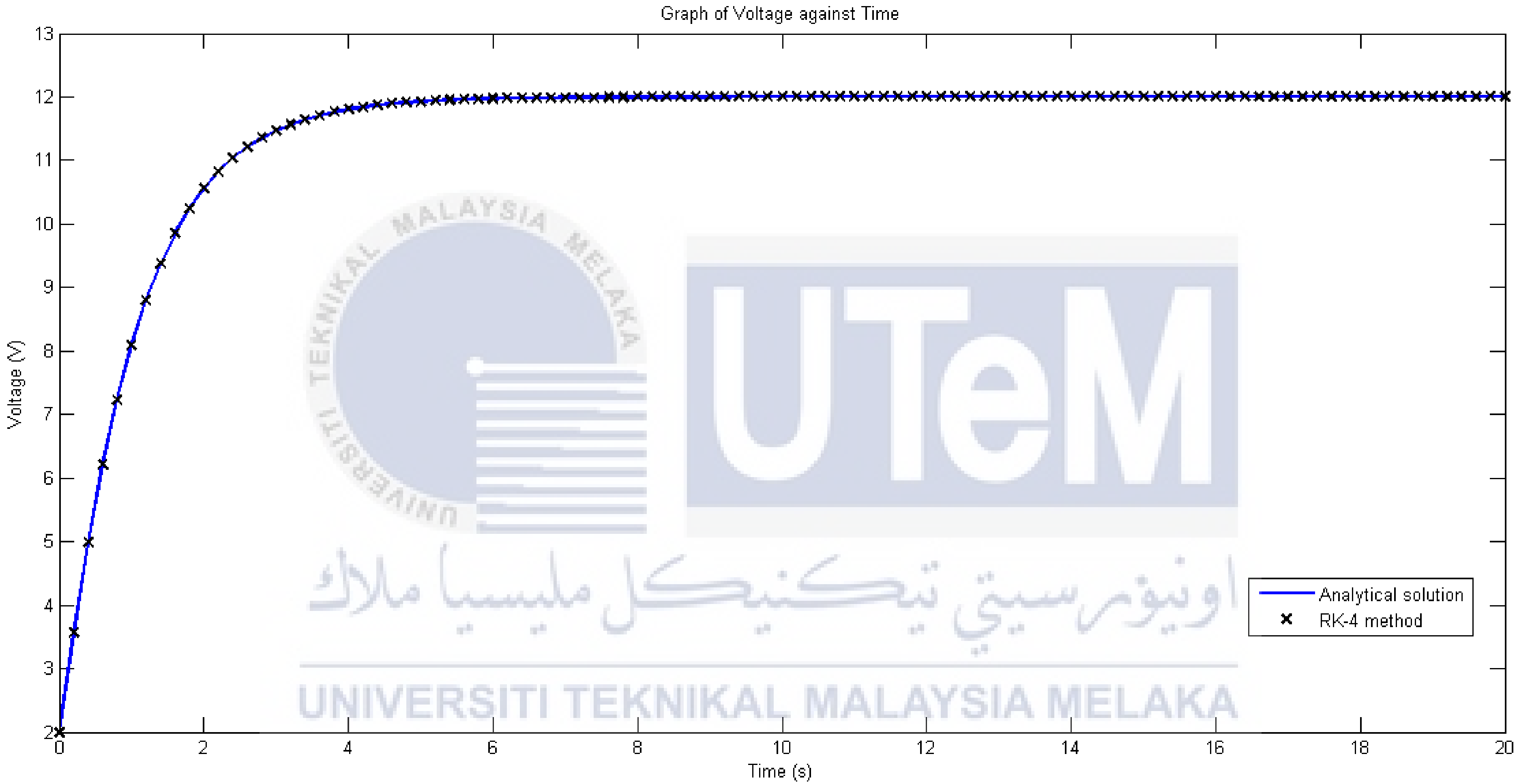

Figure 4.5: Transient Analysis of RLC circuit using analytical solution and RK-4 method ($R1$=5Ω, L=1H, C=0.25F)

Table 4.9: The minimum error, maximum error and average error between analytical solution and RK-4 method ($R1$=5Ω, L=1H, C=0.25F).

Error	Numerical Method
	RK-4 method
Minimum	0.00%
Maximum	0.04%
Average	0.00%

Figure 4.5 shows the transient analysis of RLC circuit using analytical solution and RK-4 method where the resistance, $R1$ is 5Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and Fourth-order Runge Kutta method are calculated and shown in Table 4.9.

A computed point is taken at 1s from analytical solution and RK-4 method in Figure 4.5 in order to determine the accuracy of Fourth-order Runge-Kutta method on transient analysis. According to the Figure 4.5, the voltage of RLC circuit obtained from the analytical solution at 1s is 8.0884V meanwhile the voltage obtained from the Fourth-order Runge-Kutta method is 8.0884V. As reported in results (APPENDIX C), the percentage of error in transient analysis using RK-4 method is 0.00%.

From Table 4.9, the minimum error for transient analysis using Fourth-order Runge-Kutta method is 0.00% due to same initial point of the entire line graphs which the initial points of analytical solution and RK-4 method are 2 V. By using Fourth-order Runge-Kutta method as a numerical simulation on transient analysis of electrical circuit, the maximum error and the average error obtained are about 0.04% and 0.00% respectively. Since the error obtained by RK-4 method is minute, RK-4 method is considered as a high accuracy numerical method which the accuracy is nearly 99.96% for this case.

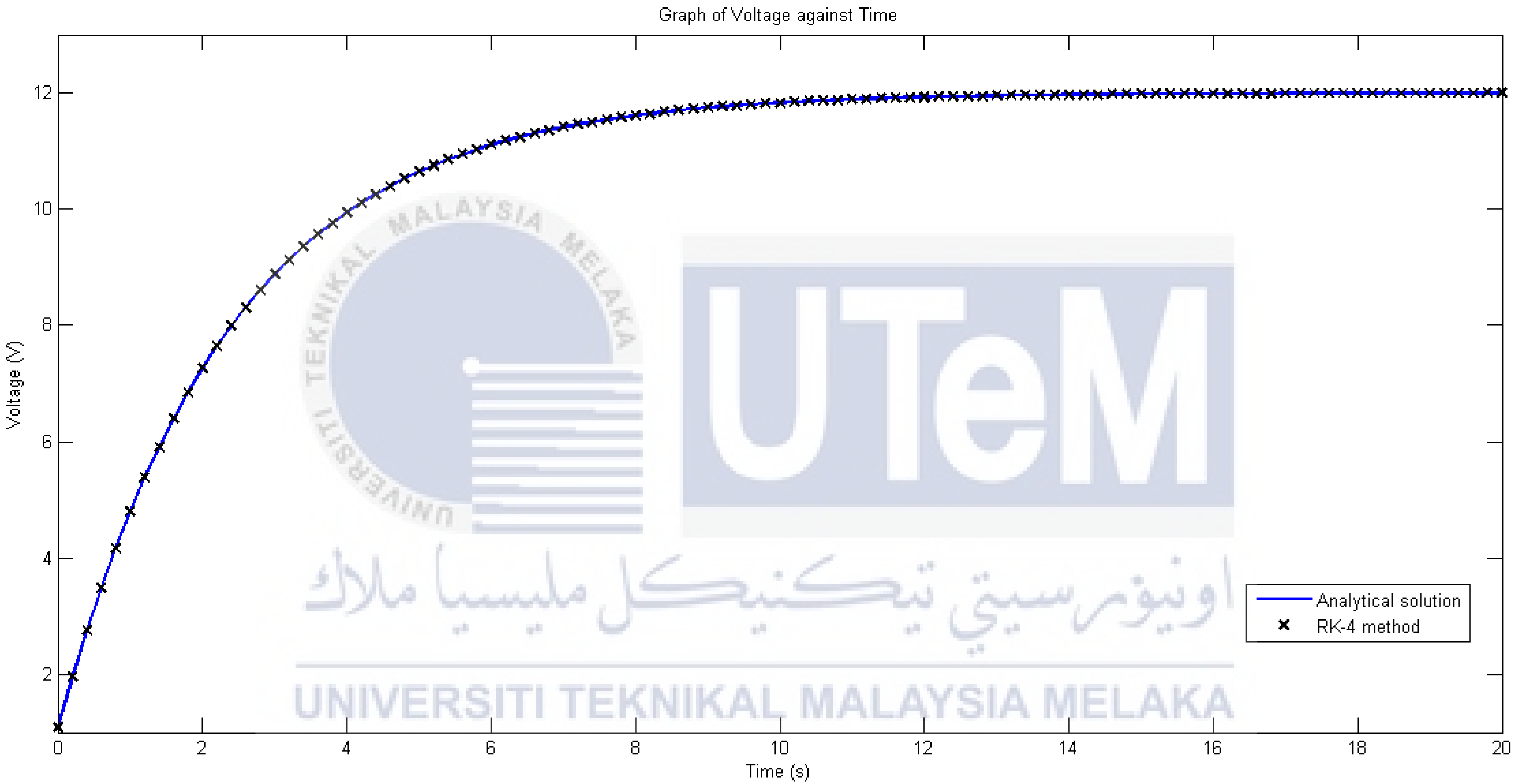

Figure 4.6: Transient Analysis of RLC circuit using analytical solution and RK-4 method ($R1$=10Ω, L=1H, C=0.25F)

Table 4.10: The minimum error, maximum error and average error between analytical solution and RK-4 method ($R1$=10Ω, L=1H, C=0.25F).

Error	Numerical Method
	RK-4 method
Minimum	0.00%
Maximum	0.17%
Average	0.00%

Figure 4.6 shows the transient analysis of RLC circuit using analytical solution and RK-4 method where the resistance, $R1$ is 10Ω, inductance, L is 1H and capacitance, C is 0.25F. The minimum error, maximum error and average error between analytical solution and Fourth-order Runge Kutta method are calculated and shown in Table 4.10.

A computed point is taken at 1s from analytical solution and RK-4 method in Figure 4.6 in order to determine the accuracy of Fourth-order Runge-Kutta method on transient analysis. According to the Figure 4.6, the voltage of RLC circuit obtained from the analytical solution at 1s is 4.7999V meanwhile the voltage obtained from the Fourth-order Runge-Kutta method is 4.8001V. As reported in results (APPENDIX C), the percentage of error in transient analysis using RK-4 method is 0.00%.

From Table 4.10, the minimum error for transient analysis using Fourth-order Runge-Kutta method is 0.00% because of both line graphs have the same initial point which the initial points of analytical solution and RK-4 method are nearly 1.0909 V. By using Fourth-order Runge-Kutta method as a numerical simulation on transient analysis of electrical circuit, the maximum error and the average error obtained are about 0.17% and 0.00% respectively. Since RK-4 method has a minute maximum error, RK-4 method is considered as a high accuracy numerical method which the accuracy is nearly 99.83% for this case.

4.2.2 Comparison of Fourth-order Runge-Kutta method with different step size, *h*

In the experiment, a series RLC circuit is set up as shown in Figure 3.2. In order to determine the relationship between Fourth-order Runge-Kutta method and step size, the experiment is carried out using Fourth-order Runge-Kutta method with different step sizes, h. In this part, the RLC circuit with underdamped response is utilised. The parameter of electrical elements of RLC circuit are fixed and shown in the Table 4.11. The initial voltage and initial current of the RLC circuit with resistance, *R1* are 6V and 6A respectively. The initial voltage and initial current are calculated by using the Ohm's law stated as equation (2.1) and the calculations are done as shown in APPENDIX A.

Table 4.11: The values of the electrical elements of series RLC circuit.

Element	Value
DC voltage source	12 V
Resistor, *R1*	1 Ω
Resistor, *R2*	1 Ω
Inductor, *L*	1 H
Capacitor, *C*	0.25 F

In this part, the experiment is repeated three times by considering 0.5, 0.2 and 0.1 as the time step size, *h*. The total time for the transient analysis of RLC circuit is set to 20s. Since number of steps, *n* is used in MATLAB numerical code, the time step size is changed into the form of number of steps, n while doing the experiment. The number of steps for time step size of 0.5, 0.2 and 0.1 are 40, 100 and 200 respectively.

Transient analysis of RLC circuit using analytical solution and RK-4 method with step size of 0.5, 0.2 and 0.1 are shown in Figure 4.7, Figure 4.8 and Figure 4.9 correspondingly. Table 4.12 shows the minimum error, maximum error and average error between analytical solution and Fourth-order Runge-Kutta method with step size of 0.5, 0.2 and 0.1.

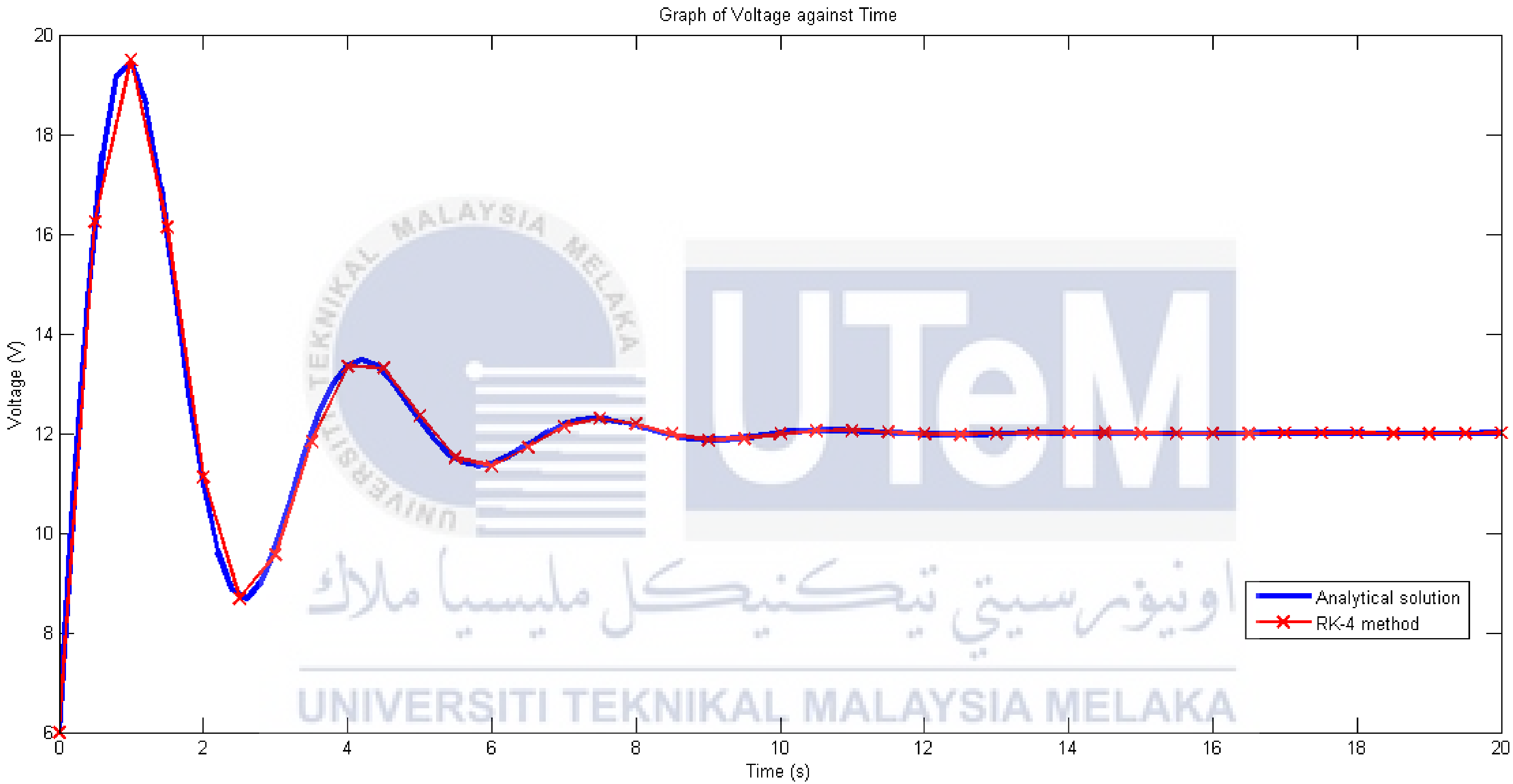

Figure 4.7: Transient Analysis of RLC circuit using analytical solution and RK-4 method (time step size, *h*=0.5 or number of steps, *n*=40)

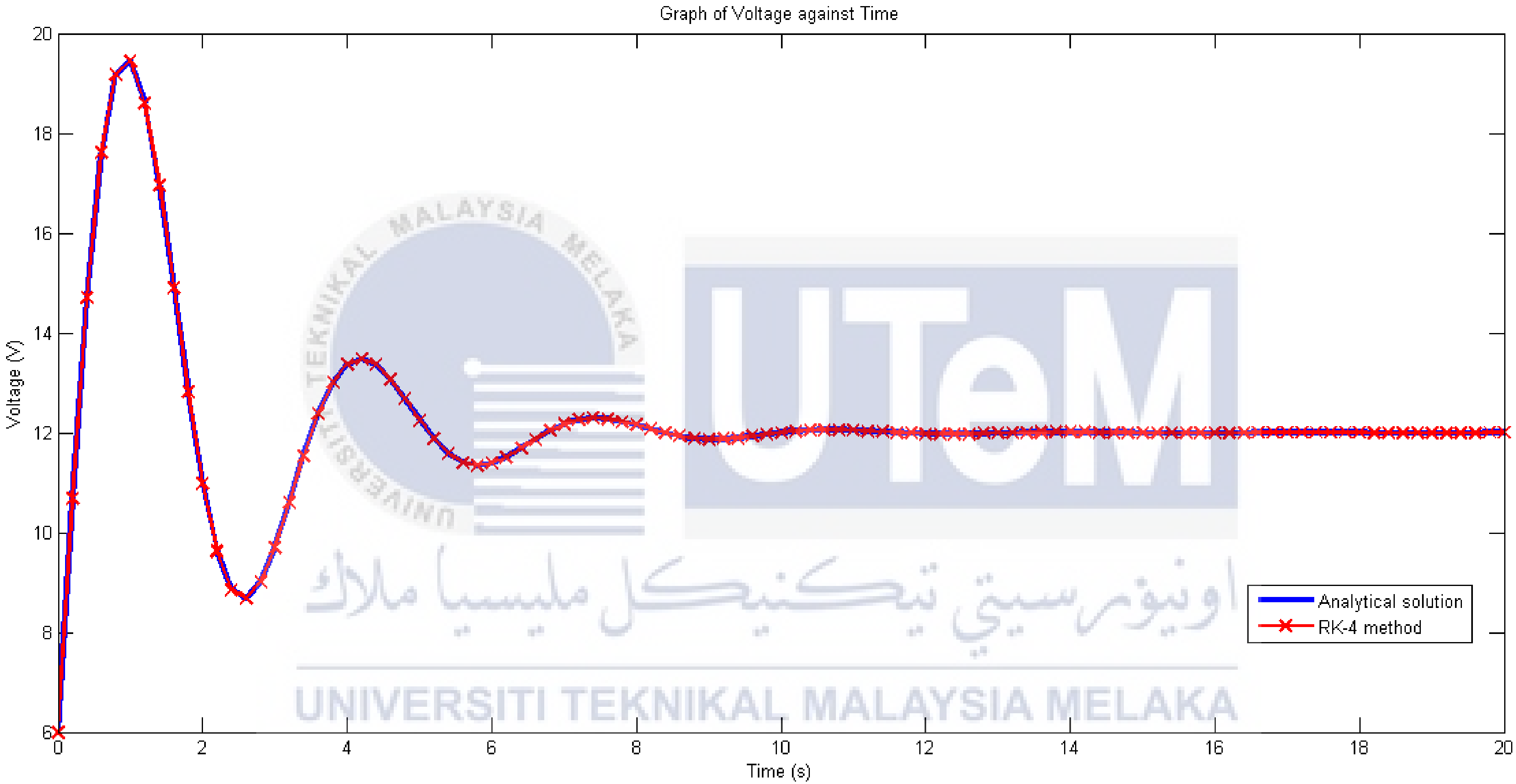

Figure 4.8: Transient Analysis of RLC circuit using analytical solution and RK-4 method (time step size, *h*=0.2 or number of steps, *n*=100)

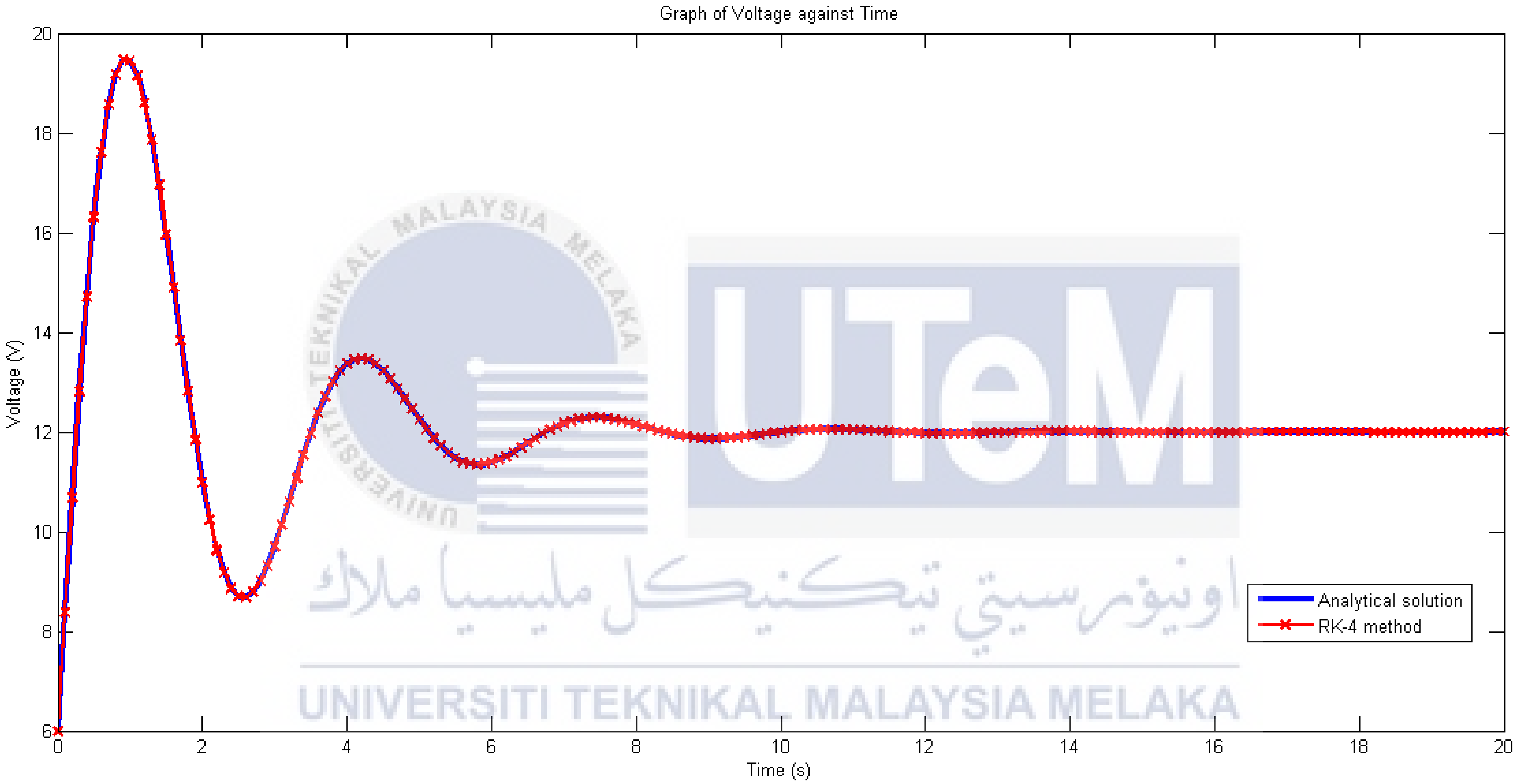

Figure 4.9: Transient Analysis of RLC circuit using analytical solution and RK-4 method (time step size, *h*=0.1 or number of steps, *n*=200)

Table 4.12: The minimum error, maximum error and average error between analytical solution and RK-4 method with step size of 0.5, 0.2 and 0.1.

Error	**Time Step Size, h**		
	0.5	**0.2**	**0.1**
Minimum	0.00%	0.00%	0.00%
Maximum	1.50%	0.04%	0.00%
Average	0.25%	0.01%	0.00%

According to Table 4.12, the minimum errors for all three step sizes are 0.00% which indicated that there is no difference between the computed points of analytical solution and all the three numerical techniques. The minimum errors for all step sizes are 0.00% because of the initial point of the entire line graphs are on the same point which is 6V. The maximum error of transient analysis using Fourth-order Runge-Kutta method with step size of 0.5 is 1.50% and the average error is 0.25%. By using step size of 0.2 in RK-4 method, the maximum error is 0.04% and the average error is 0.01%. As seen from Figure 4.9 and Table 4.12, Fourth-order Runge-Kutta method with step size of 0.1 has the lowest maximum error and average error which are 0.00%.

A computed point is taken at 1s from each line graph in Figure 4.7, Figure 4.8 and Figure 4.9 in order to compare the accuracy of the RK-4 method with different step sizes on transient analysis. From Figure 4.7, Figure 4.8 and Figure 4.9, the voltage of RLC circuit gained from the analytical solution at 1s is 19.4439V. According to Figure 4.7, with the step size of 0.5, the voltage gained from RK-4 method is 19.4902V which has the highest percentage of error of 0.24%. Based on Figure 4.8, with the step size of 0.2, the voltage gained from RK-4 method is 19.4464V and has a percentage of error of 0.01%. Based on Figure 4.9, with the step size of 0.1, the voltage gained from Fourth-order Runge-Kutta method is 19.4439V and the percentage of error is about 0.00%.

As reported in results (APPENDIX C), it is clearly shown that when a computed point is taken at 1s from each figure, step size of 0.5 has the worst accuracy of numerical simulation on the transient analysis of RLC circuit which is about 99.76%. The accuracy of the numerical simulation can be improved by decreasing the time step size or increasing the number of steps. In this research, the time step size is decreased to 0.2 and 0.1 in order to improve the accuracy of RK-4 method. When the time step size is decreased to 0.2, the error between analytical solution and RK-4 method is reduced about 0.23%. The accuracy of numerical simulation on the transient analysis of the RLC circuit has the most improvement which is approximately 0.24% when the time step size is decreased to 0.1. As the number of steps increases or the time step size decreases, the accuracy of the numerical simulation on transient analysis using Fourth-order Runge-Kutta method increases.

CHAPTER 5

CONCLUSION

This chapter would discuss the overall project's progress and brief analysis of result with the potential of improvement. Besides, the conclusion emphasizes over the significant part of the experiment which is the transient analysis using Runge-Kutta method. The last portion would be recommendation works to be done after completing the project.

5.1 Conclusion

The transient analysis of electrical circuit is analysed using analytical method and Fourth-order Runge-Kutta method. With the usage of MATLAB, the process of obtaining results of transient analysis can be done systematically and convenient. The transient analysis is only done for simple series RLC circuit in three conditions which are underdamped response, critically damped response and overdamped response. All of the responses are simulated using MATLAB and visualised in a graphical form. All the related data are transferred to Microsoft Excel in order to analyse the accuracy of the numerical simulations. In the first numerical simulation, three numerical methods are used to run the numerical simulation and the results are compared with the analytical solution. Throughout the numerical simulations, Fourth-order Runge-Kutta method has the highest degree of accuracy which is up to 99.99%. In the second numerical simulation, transient analysis of

RLC circuit is analysed using Fourth-order Runge-Kutta method with different time step size, h. In this simulation, the time step size is decreased gradually from 0.5 to 0.1 in order to determine the relationship of the accuracy of RK-4 method and time step size. By using step size of 0.1 in Fourth-order Runge-Kutta method, the accuracy of this method is up to 100%. As the step size decimates, the absolute error also gets decimated. In other word, the accuracy of Fourth-order Runge-Kutta method can be improved by decreasing the step size, h or increasing the number of steps, n. As a result, Fourth-order Runge-Kutta method is a suitable method for solving transient analysis of electric circuit due to its high degree of accuracy and efficiency in solving second-order differential equations.

5.2 Recommendation

There is more improvement needed for transient analysis of electrical circuit. Fourth-order Runge-Kutta method could be implemented to solve a higher order of ordinary differential equations such as third-order differential equations or fourth-order differential equations. Besides, transient analysis could be conducted on more complex design electrical circuits. In physical application and testing, a model based design with actual hardware could be built.

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APPENDIX A

Analysis of RLC circuit

Condition 1: Underdamped Response

1. R1 = 1Ω

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 acts like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{1 + 1} = 6\,A$$

$$v(0) = 1i(0) = 1(6) = 6\,V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is underdamped response, the total response of this condition is shown as equation (3.13).

$$v(t) = V_s + e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{1}{2 \times 1} = 0.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, ω_d, A_1 and A_2 are determined. The damping frequency, ω_d is determined by using equation (3.11).

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} = \sqrt{2^2 - 0.5^2} = 1.9365$$

The value of ω_d is then substituted into equation (3.13) to determine the constants A_1 and A_2.

$$v(t) = 12 + e^{-0.5t}(A_1 \cos 1.9365t + A_2 \sin 1.9365t)$$

When $t = 0$,

$$v(0) = 6 = 12 + A_1$$

$$A_1 = -6$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 6$$

$$\frac{dv(0)}{dt} = \frac{6}{0.25} = 24$$

In order to find $\frac{dv}{dt}$, the product rule of differential equation is used. Hence,

$$\frac{dv}{dt} = e^{-0.5t}(-1.9365A_1 \sin 1.9365t + 1.9365A_2 \cos 1.9365t) - 0.5e^{-0.5t}(A_1 \cos 1.9365t + A_2 \sin 1.9365t)$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 24 = (-0 + 1.9365A_2) - 0.5(A_1 + 0)$$

$$-0.5A_1 + 1.9365A_2 = 24$$

Substitute $A_1 = -6$ into the equation above,

$$-0.5(-6) + 1.9365A_2 = 24$$

$$3 + 1.9365A_2 = 24$$

$$A_2 = 10.8443$$

Therefore, the complete solution is

$$v(t) = 12 + e^{-0.5t}(-6 \cos 1.9365t + 10.8443 \sin 1.9365t)\ V$$

2. <u>R1 = 2Ω</u>

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 acts like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{2 + 1} = 4\,A$$

$$v(0) = 1i(0) = 1(4) = 4\,V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is underdamped response, the total response of this condition is shown as equation (3.13).

$$v(t) = V_s + e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{2}{2 \times 1} = 1$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, ω_d, A_1 and A_2 are determined. The damping frequency, ω_d is determined by using equation (3.11).

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} = \sqrt{2^2 - 1^2} = 1.7321$$

The value of ω_d is then substituted into equation (3.13) to determine the constants A_1 and A_2.

$$v(t) = 12 + e^{-t}(A_1 \cos 1.7321t + A_2 \sin 1.7321t)$$

When $t = 0$,

$$v(0) = 4 = 12 + A_1$$

$$A_1 = -8$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 4$$

$$\frac{dv(0)}{dt} = \frac{4}{0.25} = 16$$

In order to find $\frac{dv}{dt}$, the product rule of differential equation is used. Hence,

$$\frac{dv}{dt} = e^{-t}(-1.7321A_1 \sin 1.7321t + 1.7321A_2 \cos 1.7321t) - e^{-t}(A_1 \cos 1.7321t + A_2 \sin 1.7321t)$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 16 = (-0 + 1.7321A_2) - (A_1 + 0)$$

$$-A_1 + 1.7321A_2 = 16$$

Substitute $A_1 = -8$ into the equation above,

$$-8 + 1.7321A_2 = 16$$

$$A_2 = 4.6187$$

Therefore, the complete solution is

$$v(t) = 12 + e^{-t}(-8 \cos 1.7321t + 4.6187 \sin 1.7321t)\ V$$

3. R1 = 3Ω

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 acts like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{3 + 1} = 3\ A$$

$$v(0) = 1i(0) = 1(3) = 3\ V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is underdamped response, the total response of this condition is shown as equation (3.13).

$$v(t) = V_s + e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{3}{2 \times 1} = 1.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, ω_d, A_1 and A_2 are determined. The damping frequency, ω_d is determined by using equation (3.11).

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} = \sqrt{2^2 - 1.5^2} = 1.3229$$

The value of ω_d is then substituted into equation (3.13) to determine the constants A_1 and A_2.

$$v(t) = 12 + e^{-1.5t}(A_1 \cos 1.3229t + A_2 \sin 1.3229t)$$

When $t = 0$,

$$v(0) = 3 = 12 + A_1$$

$$A_1 = -9$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 3$$

$$\frac{dv(0)}{dt} = \frac{3}{0.25} = 12$$

In order to find $\frac{dv}{dt}$, the product rule of differential equation is used. Hence,

$$\frac{dv}{dt} = e^{-1.5t}(-1.3229A_1 \sin 1.3229t + 1.3229A_2 \cos 1.3229t) - 1.5e^{-1.5t}(A_1 \cos 1.3229t + A_2 \sin 1.3229t)$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 12 = (-0 + 1.3229A_2) - 1.5(A_1 + 0)$$

$$-1.5A_1 + 1.3229A_2 = 12$$

Substitute $A_1 = -9$ into the equation above,

$$-1.5(-9) + 1.3229A_2 = 12$$

$$13.5 + 1.9365A_2 = 12$$

$$A_2 = -1.1339$$

Therefore, the complete solution is

$$v(t) = 12 + e^{-1.5t}(-9 \cos 1.3229t - 1.1339 \sin 1.3229t)\ V$$

Condition 2: Critically damped Response

R1 = 4Ω

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 behaves like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{4 + 1} = 2.4\,A$$

$$v(0) = 1i(0) = 1(2.4) = 2.4\,V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is critically damped response, the total response of this condition is shown as equation (3.9).

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t}$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{4}{2 \times 1} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, the constants A_1 and A_2 are determined by substituting the source voltage, V_s and the damping factor, α into the equation above.

$$v(t) = 12 + (A_1 + A_2 t)e^{-2t}$$

When $t = 0$,

$$v(0) = 2.4 = 12 + A_1$$

$$A_1 = -9.6$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 2.4$$

$$\frac{dv(0)}{dt} = \frac{2.4}{0.25} = 9.6$$

In order to find $\frac{dv}{dt}$, the product rule of differential equation is used. Hence,

$$\frac{dv}{dt} = e^{-2t}(A_2) - 2e^{-2t}(A_1 + A_2 t)$$

$$\frac{dv}{dt} = e^{-2t}(-2A_1 + A_2 - 2A_2 t)$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 9.6 = -2A_1 + A_2$$

$$-2A_1 + A_2 = 9.6$$

Substitute $A_1 = -9.6$ into the equation above,

$$-2(-9.6) + A_2 = 9.6$$

$$A_2 = -9.6$$

Therefore, the complete solution is

$$v(t) = 12 + (-9.6 - 9.6t)e^{-2t}\ V$$

and hence,

$$v(t) = 12 - 9.6(1 + t)e^{-2t}\ V$$

Condition 3: Overdamped Response

A. <u>R1 = 5Ω</u>

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 behaves like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{5+1} = 2\ A$$

$$v(0) = 1i(0) = 1(2) = 2\ V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is underdamped response, the total response of this condition is shown as equation (3.7).

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, the roots of the characteristic equation, $s_{1,2}$ and the constants A_1 and A_2 are determined. The roots of the characteristic equation, s_1 and s_2 are determined by using equation (3.4).

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} = -2.5 \pm \sqrt{2.5^2 - 2^2} = -2.5 \pm 1.5$$

$$s_1 = -1\ and\ s_2 = -4$$

The values of the roots, s_1 and s_2 are then substituted into equation (3.7) to determine the constants A_1 and A_2.

$$v(t) = 12 + A_1 e^{-t} + A_2 e^{-4t}$$

When $t = 0$,

$$v(0) = 2 = 12 + A_1 + A_2$$

$$A_1 + A_2 = -10$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 2$$

$$\frac{dv(0)}{dt} = \frac{2}{0.25} = 8$$

In order to determine the constants A_1 and A_2, the total response, $v(t)$ is differentiated with respect to time.

Hence,

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t}$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 8 = -A_1 - 4A_2$$

$$-A_1 - 4A_2 = 8$$

Substitute $A_1 = -10 - A_2$ into the equation above,

$$-(-10 - A_2) - 4A_2 = 8$$

$$10 - 3A_2 = 8$$

$$A_2 = \frac{2}{3}$$

Substitute $A_2 = \frac{2}{3}$ into the equation of $A_1 + A_2 = -10$,

$$A_1 + \frac{2}{3} = -10$$

$$A_1 = -\frac{32}{3}$$

Therefore, the complete solution is

$$v(t) = 12 - \frac{32}{3}e^{-t} + \frac{2}{3}e^{-4t}\ V$$

B. R1 = 10Ω

For $t < 0$, the capacitor of the circuit as shown in Figure 3.2 behaves like an open circuit whereas the inductor of the same circuit behaves like a short circuit. The initial condition of the circuit is determined by calculating the initial current passes through the inductor, $i(0)$ and the initial voltage across the capacitor, $v(0)$. Since the capacitor is parallel to the 1Ω resistor, the initial voltage across the capacitor is equivalent to the voltage across the 1Ω resistor.

$$i(0) = \frac{V}{R1 + R2} = \frac{12}{10 + 1} = 1.0909\ A$$

$$v(0) = 1i(0) = 1(1.0909) = 1.0909\ V$$

For $t > 0$, the 1Ω resistor is disconnected because the switch S1 is opened. There is a series RLC circuit with a DC voltage source remains in the circuit. Since the response is underdamped response, the total response of this condition is shown as equation (3.7).

$$v(t) = V_s + A_1e^{s_1t} + A_2e^{s_2t}$$

The damping factor, α and undamped natural frequency, ω_0 are shown as following:

$$\alpha = \frac{R}{2L} = \frac{10}{2 \times 1} = 5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

In order to complete the total response, the roots of the characteristic equation, $s_{1,2}$ and the constants A_1 and A_2 are determined. The roots of the characteristic equation, s_1 and s_2 are determined by using equation (3.4).

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} = -5 \pm \sqrt{5^2 - 2^2} = -5 \pm 4.5826$$

$$s_1 = -0.4174 \; and \; s_2 = -9.5826$$

The values of the roots, s_1 and s_2 are then substituted into equation (3.7) to determine the constants A_1 and A_2.

$$v(t) = 12 + A_1 e^{-0.4174t} + A_2 e^{-9.5826t}$$

When $t = 0$,

$$v(0) = 1.0909 = 12 + A_1 + A_2$$

$$A_1 + A_2 = -10.9091$$

Since $i = C\frac{dv}{dt}$,

$$i(0) = C\frac{dv(0)}{dt} = 1.0909$$

$$\frac{dv(0)}{dt} = \frac{1.0909}{0.25} = 4.3636$$

In order to determine the constants A_1 and A_2, the total response, $v(t)$ is differentiated with respect to time.

Hence,

$$\frac{dv}{dt} = -0.4174A_1 e^{-0.4174t} - 9.5826A_2 e^{-9.5826t}$$

When $t = 0$,

$$\frac{dv(0)}{dt} = 4.3636 = -0.4174A_1 - 9.5826A_2$$

$$-0.4174A_1 - 9.5826A_2 = 4.3636$$

Substitute $A_1 = -10.9091 - A_2$ into the equation above,

$$-0.4174(-10.9091 - A_2) - 9.5826A_2 = 4.3636$$

$$4.5535 - 9.1652A_2 = 4.3636$$

$$A_2 = 0.02072$$

Substitute $A_2 = 0.02072$ into the equation of $A_1 + A_2 = -10.9091$,

$$A_1 + 0.02072 = -10.9091$$

$$A_1 = -10.9298$$

Therefore, the complete solution is

$$v(t) = 12 - 10.9298e^{-0.4174t} + 0.02072e^{-9.5826t}\ V$$

APPENDIX B

MATLAB Code

Code for Euler method:

```
function y=euler_1_1ohm(y0,z0,x0,x1)
n=100;                                % n is number of step size
h=(x1-x0)/n;                          % h is step size
x=x0:h:x1;                            % range of x is from x0 to x1

y(1)=y0;                              % Initialise y
z(1)=z0;                              % Initialise z

f=inline('(12-y-z)/1','x','y','z');   % function f = (V-R*I(t)-V(t))/L
g=inline('y/0.25','x','y','z');       % function g = I(t)/C

for i=1:n;
    y(i+1)=y(i)+h*(feval(f,x(i),y(i),z(i))); % Euler Formula
    z(i+1)=z(i)+h*(feval(g,x(i),y(i),z(i)));
end;

plot(x,z,'m')                         % plot graph of voltage against time
```

Code for Heun’s method:

```
function y=heun_1_1ohm(y0,y1,x0,x1)
n=100;                                % n is number of step size
h=(x1-x0)/n;                          % h is step size
x=x0:h:x1;                            % range of x is from x0 to x1

y(1)=y0;                              % Initialise y
z(1)=z0;                              % Initialise z

f=inline('(12-y-z)/1','x','y','z');   % function f = (V-R*I(t)-V(t))/L
g=inline('y/0.25','x','y','z');       % function g = I(t)/C

for i=1:n;
    K1=(feval(f,x(i),y(i),z(i)));     % Heun’s Formula
    L1=(feval(g,x(i),y(i),z(i)));
    K2=(feval(f,x(i)+h,y(i)+h*K1,z(i)+h*L1));
    L2=(feval(g,x(i)+h,y(i)+h*K1,z(i)+h*L1));

    y(i+1)=y(i)+(h/2)*(K1+K2);
    z(i+1)=z(i)+(h/2)*(L1+L2);
end;

plot(x,z,'r')                         % plot graph of voltage against time
```

Code for Fourth-order Runge-Kutta method:

```
function y=RK4_1_1ohm(y0,y1,x0,x1)
n=100;                              % n is number of step size
h=(x1-x0)/n;                        % h is step size
x=x0:h:x1;                          % range of x is from x0 to x1

y(1)=y0;                            % Initialise y
z(1)=z0;                            % Initialise z

f=inline('(12-y-z)/1','x','y','z'); % function f = (V-R*I(t)-V(t))/L
g=inline('y/0.25','x','y','z');     % function g = I(t)/C

for i=1:n;
    K1=(feval(f,x(i),y(i),z(i)));   % RK-4 Formula
    L1=(feval(g,x(i),y(i),z(i)));
    K2=(feval(f,x(i)+h/2,y(i)+K1/2*h,z(i)+L1/2*h));
    L2=(feval(g,x(i)+h/2,y(i)+K1/2*h,z(i)+L1/2*h));
    K3=(feval(f,x(i)+h/2,y(i)+K2/2*h,z(i)+L2/2*h));
    L3=(feval(g,x(i)+h/2,y(i)+K2/2*h,z(i)+L2/2*h));
    K4=(feval(f,x(i)+h,y(i)+K3*h,z(i)+L3*h));
    L4=(feval(g,x(i)+h,y(i)+K3*h,z(i)+L3*h));

    y(i+1)=y(i)+(h/6)*(K1+2*K2+2*K3+K4);
    z(i+1)=z(i)+(h/6)*(L1+2*L2+2*L3+L4);
end;

plot(x,z,'kx')                      % plot graph of voltage against time
```

APPENDIX C

Tabulated Data for Transient Analysis of Electrical Circuit

A. Numerical Simulation 1: Comparison between Analytical Solution and Numerical Methods

Condition 1: Underdamped Response

R1 = 1Ω

Time (s)	Analytical solution	Euler method		Heun's method		RK-4 method	
	Computed Point	Computed Point	error	Computed Point	error	Computed Point	error
0	6.0000	6.0000	0.00%	6.0000	0.00%	6.0000	0.00%
0.2	10.6791	10.8000	1.13%	10.8000	1.13%	10.6784	0.01%
0.4	14.6990	15.6000	6.13%	14.8704	1.17%	14.6984	0.00%
0.6	17.6041	19.6320	11.52%	17.7373	0.76%	17.6041	0.00%
0.8	19.1806	22.2816	16.17%	19.1978	0.09%	19.1818	0.01%
1	19.4438	23.1802	19.22%	19.2988	0.75%	19.4464	0.01%
1.2	18.5942	22.2540	19.68%	18.2830	1.67%	18.5977	0.02%
1.4	16.9550	19.7242	16.33%	16.5147	2.60%	16.9590	0.02%
1.6	14.9039	16.0597	7.76%	14.4027	3.36%	14.9077	0.03%
1.8	12.8090	11.8923	7.16%	12.3298	3.74%	12.8121	0.02%
2	10.9781	7.9088	27.96%	10.6003	3.44%	10.9799	0.02%
2.2	9.6254	4.7392	50.76%	9.4092	2.25%	9.6255	0.00%
2.4	8.8576	2.8581	67.73%	8.8336	0.27%	8.8561	0.02%
2.6	8.6787	2.5150	71.02%	8.8434	1.90%	8.6759	0.03%
2.8	9.0075	3.7032	58.89%	9.3261	3.54%	9.0038	0.04%
3	9.7048	6.1714	36.41%	10.1194	4.27%	9.7010	0.04%
3.2	10.6042	9.4734	10.66%	11.0451	4.16%	10.6007	0.03%
3.4	11.5403	13.0476	13.06%	11.9389	3.45%	11.5376	0.02%
3.6	12.3725	16.3112	31.83%	12.6725	2.42%	12.3710	0.01%
3.8	13.0005	18.7545	44.26%	13.1658	1.27%	13.0003	0.00%
4	13.3716	20.0193	49.72%	13.3903	0.14%	13.3725	0.01%
4.2	13.4791	19.9504	48.01%	13.3631	0.86%	13.4810	0.01%
4.4	13.3554	18.6123	39.36%	13.1360	1.64%	13.3579	0.02%
4.6	13.0602	16.2697	24.57%	12.7812	2.14%	13.0629	0.02%
4.8	12.6669	13.3376	5.30%	12.3761	2.30%	12.6693	0.02%
5	12.2494	10.3088	15.84%	11.9913	2.11%	12.2512	0.02%
5.2	11.8719	7.6718	35.38%	11.6807	1.61%	11.8729	0.01%
5.4	11.5811	5.8327	49.64%	11.4771	0.90%	11.5814	0.00%
5.6	11.4030	5.0540	55.68%	11.3907	0.11%	11.4025	0.00%
5.8	11.3426	5.4178	52.24%	11.4123	0.61%	11.3415	0.01%

6	11.3873	6.8202	40.11%	11.5182	1.15%	11.3858	0.01%
6.2	11.5115	8.9952	21.86%	11.6765	1.43%	11.5099	0.01%
6.4	11.6831	11.5641	1.02%	11.8535	1.46%	11.6816	0.01%
6.6	11.8689	14.0999	18.80%	12.0189	1.26%	11.8678	0.01%
6.8	12.0398	16.1983	34.54%	12.1501	0.92%	12.0392	0.01%
7	12.1741	17.5410	44.09%	12.2339	0.49%	12.1739	0.00%
7.2	12.2591	17.9435	46.37%	12.2666	0.06%	12.2593	0.00%
7.4	12.2916	17.3789	41.39%	12.2530	0.31%	12.2922	0.00%
7.6	12.2765	15.9763	30.14%	12.2039	0.59%	12.2773	0.01%
7.8	12.2245	13.9935	14.47%	12.1335	0.74%	12.2255	0.01%
8	12.1499	11.7712	3.12%	12.0563	0.77%	12.1507	0.01%
8.2	12.0673	9.6743	19.83%	11.9853	0.68%	12.0680	0.01%
8.4	11.9901	8.0334	33.00%	11.9300	0.50%	11.9905	0.00%
8.6	11.9283	7.0928	40.54%	11.8957	0.27%	11.9284	0.00%
8.8	11.8879	6.9750	41.33%	11.8836	0.04%	11.8878	0.00%
9	11.8709	7.6659	35.42%	11.8913	0.17%	11.8706	0.00%
9.2	11.8755	9.0226	24.02%	11.9138	0.32%	11.8750	0.00%
9.4	11.8970	10.8014	9.21%	11.9451	0.40%	11.8965	0.00%
9.6	11.9294	12.7009	6.47%	11.9787	0.41%	11.9289	0.00%
9.8	11.9660	14.4122	20.44%	12.0091	0.36%	11.9656	0.00%
10	12.0008	15.6691	30.57%	12.0324	0.26%	12.0006	0.00%
10.2	12.0292	16.2887	35.41%	12.0464	0.14%	12.0292	0.00%
10.4	12.0483	16.1973	34.44%	12.0508	0.02%	12.0484	0.00%
10.6	12.0570	15.4380	28.04%	12.0467	0.09%	12.0572	0.00%
10.8	12.0560	14.1590	17.44%	12.0363	0.16%	12.0562	0.00%
11	12.0471	12.5857	4.47%	12.0225	0.20%	12.0474	0.00%
11.2	12.0331	10.9816	8.74%	12.0079	0.21%	12.0334	0.00%
11.4	12.0169	9.6046	20.07%	11.9949	0.18%	12.0171	0.00%
11.6	12.0012	8.6660	27.79%	11.9851	0.13%	12.0014	0.00%
11.8	11.9882	8.2983	30.78%	11.9794	0.07%	11.9883	0.00%
12	11.9793	8.5377	28.73%	11.9779	0.01%	11.9792	0.00%
12.2	11.9749	9.3214	22.16%	11.9800	0.04%	11.9748	0.00%
12.4	11.9749	10.5023	12.30%	11.9847	0.08%	11.9748	0.00%
12.6	11.9785	11.8757	0.86%	11.9908	0.10%	11.9784	0.00%
12.8	11.9845	13.2140	10.26%	11.9972	0.11%	11.9844	0.00%
13	11.9917	14.3045	19.29%	12.0027	0.09%	11.9916	0.00%
13.2	11.9987	14.9827	24.87%	12.0068	0.07%	11.9986	0.00%
13.4	12.0047	15.1565	26.26%	12.0091	0.04%	12.0046	0.00%
13.6	12.0089	14.8184	23.40%	12.0096	0.01%	12.0089	0.00%
13.8	12.0110	14.0428	16.92%	12.0086	0.02%	12.0111	0.00%
14	12.0112	12.9714	7.99%	12.0064	0.04%	12.0113	0.00%
14.2	12.0098	11.7874	1.85%	12.0037	0.05%	12.0099	0.00%
14.4	12.0072	10.6848	11.01%	12.0010	0.05%	12.0073	0.00%
14.6	12.0041	9.8368	18.05%	11.9986	0.05%	12.0041	0.00%
14.8	12.0009	9.3687	21.93%	11.9969	0.03%	12.0009	0.00%

15	11.9982	9.3405	22.15%	11.9960	0.02%	11.9982	0.00%
15.2	11.9962	9.7388	18.82%	11.9958	0.00%	11.9962	0.00%
15.4	11.9952	10.4830	12.61%	11.9963	0.01%	11.9951	0.00%
15.6	11.9950	11.4402	4.63%	11.9973	0.02%	11.9949	0.00%
15.8	11.9955	12.4486	3.78%	11.9985	0.02%	11.9955	0.00%
16	11.9966	13.3450	11.24%	11.9997	0.03%	11.9966	0.00%
16.2	11.9980	13.9902	16.60%	12.0007	0.02%	11.9980	0.00%
16.4	11.9994	14.2913	19.10%	12.0014	0.02%	11.9994	0.00%
16.6	12.0007	14.2137	18.44%	12.0018	0.01%	12.0007	0.00%
16.8	12.0016	13.7850	14.86%	12.0018	0.00%	12.0016	0.00%
17	12.0021	13.0878	9.05%	12.0016	0.00%	12.0021	0.00%
17.2	12.0022	12.2445	2.02%	12.0011	0.01%	12.0023	0.00%
17.4	12.0020	11.3958	5.05%	12.0006	0.01%	12.0020	0.00%
17.6	12.0016	10.6777	11.03%	12.0001	0.01%	12.0016	0.00%
17.8	12.0009	10.1999	15.01%	11.9997	0.01%	12.0010	0.00%
18	12.0003	10.0293	16.42%	11.9994	0.01%	12.0003	0.00%
18.2	11.9998	10.1807	15.16%	11.9992	0.00%	11.9998	0.00%
18.4	11.9993	10.6172	11.52%	11.9992	0.00%	11.9993	0.00%
18.6	11.9991	11.2575	6.18%	11.9993	0.00%	11.9991	0.00%
18.8	11.9990	11.9910	0.07%	11.9995	0.00%	11.9990	0.00%
19	11.9991	12.6965	5.81%	11.9998	0.01%	11.9991	0.00%
19.2	11.9993	13.2624	10.53%	12.0000	0.01%	11.9993	0.00%
19.4	11.9995	13.6037	13.37%	12.0002	0.01%	11.9995	0.00%
19.6	11.9998	13.6747	13.96%	12.0003	0.00%	11.9998	0.00%
19.8	12.0001	13.4750	12.29%	12.0003	0.00%	12.0001	0.00%
20	12.0003	13.0472	8.72%	12.0003	0.00%	12.0003	0.00%

<u>R1 = 2Ω</u>

Time (s)	Analytical solution	Euler method		Heun's method		RK-4 method	
	Computed Point	Computed Point	error	Computed Point	error	Computed Point	error
0	4.0000	4.0000	0.00%	4.0000	0.00%	4.0000	0.00%
0.2	7.1232	7.2000	1.08%	7.2000	1.08%	7.1232	0.00%
0.4	9.8514	10.4000	5.57%	9.9392	0.89%	9.8518	0.00%
0.6	11.9597	13.0880	9.43%	12.0087	0.41%	11.9607	0.01%
0.8	13.3782	14.9568	11.80%	13.3618	0.12%	13.3796	0.01%
1	14.1497	15.9040	12.40%	14.0643	0.60%	14.1513	0.01%
1.2	14.3871	15.9992	11.21%	14.2465	0.98%	14.3887	0.01%
1.4	14.2356	15.4317	8.40%	14.0638	1.21%	14.2370	0.01%
1.6	13.8431	14.4514	4.39%	13.6669	1.27%	13.8441	0.01%
1.8	13.3402	13.3141	0.20%	13.1831	1.18%	13.3407	0.00%
2	12.8287	12.2395	4.59%	12.7075	0.94%	12.8287	0.00%
2.2	12.3779	11.3844	8.03%	12.3012	0.62%	12.3776	0.00%

2.4	12.0266	10.8331	9.92%	11.9948	0.26%	12.0260	0.00%
2.6	11.7877	10.6008	10.07%	11.7950	0.06%	11.7869	0.01%
2.8	11.6551	10.6481	8.64%	11.6918	0.32%	11.6544	0.01%
3	11.6111	10.9004	6.12%	11.6657	0.47%	11.6105	0.01%
3.2	11.6323	11.2681	3.13%	11.6935	0.53%	11.6318	0.00%
3.4	11.6943	11.6646	0.25%	11.7530	0.50%	11.6940	0.00%
3.6	11.7757	12.0196	2.07%	11.8250	0.42%	11.7755	0.00%
3.8	11.8594	12.2863	3.60%	11.8957	0.31%	11.8594	0.00%
4	11.9338	12.4432	4.27%	11.9560	0.19%	11.9339	0.00%
4.2	11.9923	12.4915	4.16%	12.0013	0.08%	11.9925	0.00%
4.4	12.0325	12.4496	3.47%	12.0308	0.01%	12.0327	0.00%
4.6	12.0552	12.3458	2.41%	12.0460	0.08%	12.0554	0.00%
4.8	12.0633	12.2116	1.23%	12.0497	0.11%	12.0635	0.00%
5	12.0604	12.0757	0.13%	12.0455	0.12%	12.0606	0.00%
5.2	12.0507	11.9604	0.75%	12.0366	0.12%	12.0508	0.00%
5.4	12.0375	11.8790	1.32%	12.0259	0.10%	12.0376	0.00%
5.6	12.0238	11.8366	1.56%	12.0154	0.07%	12.0238	0.00%
5.8	12.0116	11.8305	1.51%	12.0064	0.04%	12.0115	0.00%
6	12.0018	11.8529	1.24%	11.9997	0.02%	12.0018	0.00%
6.2	11.9951	11.8935	0.85%	11.9954	0.00%	11.9950	0.00%
6.4	11.9912	11.9414	0.41%	11.9931	0.02%	11.9911	0.00%
6.6	11.9897	11.9872	0.02%	11.9926	0.02%	11.9897	0.00%
6.8	11.9901	12.0240	0.28%	11.9932	0.03%	11.9900	0.00%
7	11.9916	12.0482	0.47%	11.9946	0.02%	11.9916	0.00%
7.2	11.9937	12.0588	0.54%	11.9962	0.02%	11.9937	0.00%
7.4	11.9960	12.0575	0.51%	11.9977	0.01%	11.9960	0.00%
7.6	11.9980	12.0473	0.41%	11.9991	0.01%	11.9980	0.00%
7.8	11.9996	12.0320	0.27%	12.0001	0.00%	11.9996	0.00%
8	12.0007	12.0152	0.12%	12.0007	0.00%	12.0008	0.00%
8.2	12.0014	12.0000	0.01%	12.0010	0.00%	12.0014	0.00%
8.4	12.0017	11.9885	0.11%	12.0011	0.00%	12.0017	0.00%
8.6	12.0016	11.9816	0.17%	12.0010	0.01%	12.0016	0.00%
8.8	12.0014	11.9793	0.18%	12.0008	0.00%	12.0014	0.00%
9	12.0010	11.9808	0.17%	12.0006	0.00%	12.0011	0.00%
9.2	12.0007	11.9851	0.13%	12.0003	0.00%	12.0007	0.00%
9.4	12.0003	11.9907	0.08%	12.0001	0.00%	12.0003	0.00%
9.6	12.0001	11.9965	0.03%	12.0000	0.00%	12.0001	0.00%
9.8	11.9999	12.0014	0.01%	11.9999	0.00%	11.9999	0.00%
10	11.9998	12.0049	0.04%	11.9998	0.00%	11.9998	0.00%
10.2	11.9997	12.0068	0.06%	11.9998	0.00%	11.9997	0.00%
10.4	11.9997	12.0072	0.06%	11.9999	0.00%	11.9997	0.00%
10.6	11.9998	12.0063	0.05%	11.9999	0.00%	11.9998	0.00%
10.8	11.9998	12.0046	0.04%	11.9999	0.00%	11.9998	0.00%
11	11.9999	12.0026	0.02%	12.0000	0.00%	11.9999	0.00%
11.2	11.9999	12.0007	0.01%	12.0000	0.00%	11.9999	0.00%

11.4	12.0000	11.9991	0.01%	12.0000	0.00%	12.0000	0.00%
11.6	12.0000	11.9980	0.02%	12.0000	0.00%	12.0000	0.00%
11.8	12.0000	11.9975	0.02%	12.0000	0.00%	12.0000	0.00%
12	12.0000	11.9976	0.02%	12.0000	0.00%	12.0000	0.00%
12.2	12.0000	11.9980	0.02%	12.0000	0.00%	12.0000	0.00%
12.4	12.0000	11.9986	0.01%	12.0000	0.00%	12.0000	0.00%
12.6	12.0000	11.9993	0.01%	12.0000	0.00%	12.0000	0.00%
12.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13	12.0000	12.0005	0.00%	12.0000	0.00%	12.0000	0.00%
13.2	12.0000	12.0008	0.01%	12.0000	0.00%	12.0000	0.00%
13.4	12.0000	12.0009	0.01%	12.0000	0.00%	12.0000	0.00%
13.6	12.0000	12.0008	0.01%	12.0000	0.00%	12.0000	0.00%
13.8	12.0000	12.0006	0.01%	12.0000	0.00%	12.0000	0.00%
14	12.0000	12.0004	0.00%	12.0000	0.00%	12.0000	0.00%
14.2	12.0000	12.0002	0.00%	12.0000	0.00%	12.0000	0.00%
14.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14.6	12.0000	11.9998	0.00%	12.0000	0.00%	12.0000	0.00%
14.8	12.0000	11.9997	0.00%	12.0000	0.00%	12.0000	0.00%
15	12.0000	11.9997	0.00%	12.0000	0.00%	12.0000	0.00%
15.2	12.0000	11.9997	0.00%	12.0000	0.00%	12.0000	0.00%
15.4	12.0000	11.9998	0.00%	12.0000	0.00%	12.0000	0.00%
15.6	12.0000	11.9999	0.00%	12.0000	0.00%	12.0000	0.00%
15.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16.2	12.0000	12.0001	0.00%	12.0000	0.00%	12.0000	0.00%
16.4	12.0000	12.0001	0.00%	12.0000	0.00%	12.0000	0.00%
16.6	12.0000	12.0001	0.00%	12.0000	0.00%	12.0000	0.00%
16.8	12.0000	12.0001	0.00%	12.0000	0.00%	12.0000	0.00%
17	12.0000	12.0001	0.00%	12.0000	0.00%	12.0000	0.00%
17.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
20	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%

R1 = 3Ω

Time (s)	Analytical solution	Euler method		Heun's method		RK-4 method	
	Computed Point	Computed Point	error	Computed Point	error	Computed Point	error
0	3.0000	3.0000	0.00%	3.0000	0.00%	3.0000	0.00%
0.2	5.3450	5.4000	1.03%	5.4000	1.03%	5.3456	0.01%
0.4	7.4221	7.8000	5.09%	7.4736	0.69%	7.4231	0.01%
0.6	9.1056	9.8160	7.80%	9.1260	0.22%	9.1068	0.01%
0.8	10.3732	11.2944	8.88%	10.3559	0.17%	10.3743	0.01%
1	11.2620	12.2352	8.64%	11.2127	0.44%	11.2629	0.01%
1.2	11.8374	12.7244	7.49%	11.7673	0.59%	11.8381	0.01%
1.4	12.1725	12.8825	5.83%	12.0934	0.65%	12.1729	0.00%
1.6	12.3359	12.8298	4.00%	12.2579	0.63%	12.3361	0.00%
1.8	12.3857	12.6675	2.27%	12.3160	0.56%	12.3857	0.00%
2	12.3673	12.4698	0.83%	12.3098	0.46%	12.3671	0.00%
2.2	12.3135	12.2840	0.24%	12.2698	0.36%	12.3133	0.00%
2.4	12.2468	12.1344	0.92%	12.2163	0.25%	12.2466	0.00%
2.6	12.1809	12.0292	1.25%	12.1619	0.16%	12.1807	0.00%
2.8	12.1232	11.9656	1.30%	12.1135	0.08%	12.1230	0.00%
3	12.0770	11.9355	1.17%	12.0739	0.03%	12.0768	0.00%
3.2	12.0424	11.9289	0.94%	12.0439	0.01%	12.0423	0.00%
3.4	12.0184	11.9366	0.68%	12.0225	0.03%	12.0184	0.00%
3.6	12.0031	11.9511	0.43%	12.0084	0.04%	12.0030	0.00%
3.8	11.9943	11.9670	0.23%	11.9997	0.05%	11.9943	0.00%
4	11.9901	11.9812	0.07%	11.9951	0.04%	11.9901	0.00%
4.2	11.9890	11.9922	0.03%	11.9932	0.03%	11.9890	0.00%
4.4	11.9897	11.9995	0.08%	11.9930	0.03%	11.9897	0.00%
4.6	11.9913	12.0038	0.10%	11.9937	0.02%	11.9913	0.00%
4.8	11.9932	12.0055	0.10%	11.9948	0.01%	11.9933	0.00%
5	11.9951	12.0056	0.09%	11.9961	0.01%	11.9951	0.00%
5.2	11.9967	12.0048	0.07%	11.9972	0.00%	11.9967	0.00%
5.4	11.9980	12.0035	0.05%	11.9981	0.00%	11.9980	0.00%
5.6	11.9989	12.0023	0.03%	11.9988	0.00%	11.9989	0.00%
5.8	11.9995	12.0012	0.01%	11.9994	0.00%	11.9995	0.00%
6	12.0000	12.0004	0.00%	11.9997	0.00%	12.0000	0.00%
6.2	12.0002	11.9999	0.00%	12.0000	0.00%	12.0002	0.00%
6.4	12.0003	11.9996	0.01%	12.0001	0.00%	12.0003	0.00%
6.6	12.0003	11.9995	0.01%	12.0001	0.00%	12.0003	0.00%
6.8	12.0003	11.9996	0.01%	12.0002	0.00%	12.0003	0.00%
7	12.0002	11.9996	0.00%	12.0001	0.00%	12.0002	0.00%
7.2	12.0002	11.9997	0.00%	12.0001	0.00%	12.0002	0.00%
7.4	12.0001	11.9998	0.00%	12.0001	0.00%	12.0001	0.00%
7.6	12.0001	11.9999	0.00%	12.0001	0.00%	12.0001	0.00%
7.8	12.0001	12.0000	0.00%	12.0000	0.00%	12.0001	0.00%

8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
8.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
8.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
8.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
8.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
9	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
9.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
9.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
9.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
9.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
10	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
10.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
10.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
10.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
10.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
11	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
11.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
11.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
11.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
11.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
12	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
12.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
12.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
12.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
12.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
13.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
14.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
15	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
15.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
15.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
15.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
15.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
16.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%

17	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
17.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
18.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.2	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.4	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.6	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
19.8	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%
20	12.0000	12.0000	0.00%	12.0000	0.00%	12.0000	0.00%

Condition 2: Critically damped Response

<u>R1 = 4Ω</u>

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	2.4000	2.4000	0.00%
0.2	4.2779	4.2790	0.03%
0.4	5.9610	5.9623	0.02%
0.6	7.3737	7.3748	0.01%
0.8	8.5112	8.5120	0.01%
1	9.4016	9.4021	0.01%
1.2	10.0840	10.0843	0.00%
1.4	10.5989	10.5991	0.00%
1.6	10.9826	10.9826	0.00%
1.8	11.2655	11.2655	0.00%
2	11.4725	11.4724	0.00%
2.2	11.6228	11.6228	0.00%
2.4	11.7314	11.7313	0.00%
2.6	11.8093	11.8093	0.00%
2.8	11.8651	11.8650	0.00%
3	11.9048	11.9048	0.00%
3.2	11.9330	11.9330	0.00%
3.4	11.9530	11.9529	0.00%
3.6	11.9670	11.9670	0.00%

3.8	11.9769	11.9769	0.00%
4	11.9839	11.9839	0.00%
4.2	11.9888	11.9888	0.00%
4.4	11.9922	11.9922	0.00%
4.6	11.9946	11.9946	0.00%
4.8	11.9962	11.9962	0.00%
5	11.9974	11.9974	0.00%
5.2	11.9982	11.9982	0.00%
5.4	11.9987	11.9987	0.00%
5.6	11.9991	11.9991	0.00%
5.8	11.9994	11.9994	0.00%
6	11.9996	11.9996	0.00%
6.2	11.9997	11.9997	0.00%
6.4	11.9998	11.9998	0.00%
6.6	11.9999	11.9999	0.00%
6.8	11.9999	11.9999	0.00%
7	11.9999	11.9999	0.00%
7.2	12.0000	12.0000	0.00%
7.4	12.0000	12.0000	0.00%
7.6	12.0000	12.0000	0.00%
7.8	12.0000	12.0000	0.00%
8	12.0000	12.0000	0.00%
8.2	12.0000	12.0000	0.00%
8.4	12.0000	12.0000	0.00%
8.6	12.0000	12.0000	0.00%
8.8	12.0000	12.0000	0.00%
9	12.0000	12.0000	0.00%
9.2	12.0000	12.0000	0.00%
9.4	12.0000	12.0000	0.00%
9.6	12.0000	12.0000	0.00%
9.8	12.0000	12.0000	0.00%
10	12.0000	12.0000	0.00%
10.2	12.0000	12.0000	0.00%
10.4	12.0000	12.0000	0.00%
10.6	12.0000	12.0000	0.00%
10.8	12.0000	12.0000	0.00%
11	12.0000	12.0000	0.00%
11.2	12.0000	12.0000	0.00%
11.4	12.0000	12.0000	0.00%
11.6	12.0000	12.0000	0.00%
11.8	12.0000	12.0000	0.00%
12	12.0000	12.0000	0.00%
12.2	12.0000	12.0000	0.00%
12.4	12.0000	12.0000	0.00%
12.6	12.0000	12.0000	0.00%

12.8	12.0000	12.0000	0.00%
13	12.0000	12.0000	0.00%
13.2	12.0000	12.0000	0.00%
13.4	12.0000	12.0000	0.00%
13.6	12.0000	12.0000	0.00%
13.8	12.0000	12.0000	0.00%
14	12.0000	12.0000	0.00%
14.2	12.0000	12.0000	0.00%
14.4	12.0000	12.0000	0.00%
14.6	12.0000	12.0000	0.00%
14.8	12.0000	12.0000	0.00%
15	12.0000	12.0000	0.00%
15.2	12.0000	12.0000	0.00%
15.4	12.0000	12.0000	0.00%
15.6	12.0000	12.0000	0.00%
15.8	12.0000	12.0000	0.00%
16	12.0000	12.0000	0.00%
16.2	12.0000	12.0000	0.00%
16.4	12.0000	12.0000	0.00%
16.6	12.0000	12.0000	0.00%
16.8	12.0000	12.0000	0.00%
17	12.0000	12.0000	0.00%
17.2	12.0000	12.0000	0.00%
17.4	12.0000	12.0000	0.00%
17.6	12.0000	12.0000	0.00%
17.8	12.0000	12.0000	0.00%
18	12.0000	12.0000	0.00%
18.2	12.0000	12.0000	0.00%
18.4	12.0000	12.0000	0.00%
18.6	12.0000	12.0000	0.00%
18.8	12.0000	12.0000	0.00%
19	12.0000	12.0000	0.00%
19.2	12.0000	12.0000	0.00%
19.4	12.0000	12.0000	0.00%
19.6	12.0000	12.0000	0.00%
19.8	12.0000	12.0000	0.00%
20	12.0000	12.0000	0.00%

Condition 3: Overdamped Response

R1 = 5Ω

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	2.0000	2.0000	0.00%
0.2	3.5664	3.5680	0.04%
0.4	4.9845	4.9859	0.03%
0.6	6.2065	6.2074	0.01%
0.8	7.2343	7.2349	0.01%
1	8.0882	8.0884	0.00%
1.2	8.7927	8.7929	0.00%
1.4	9.3721	9.3721	0.00%
1.6	9.8475	9.8475	0.00%
1.8	10.2373	10.2373	0.00%
2	10.5566	10.5566	0.00%
2.2	10.8182	10.8182	0.00%
2.4	11.0324	11.0324	0.00%
2.6	11.2078	11.2077	0.00%
2.8	11.3514	11.3513	0.00%
3	11.4689	11.4689	0.00%
3.2	11.5652	11.5652	0.00%
3.4	11.6440	11.6440	0.00%
3.6	11.7085	11.7085	0.00%
3.8	11.7614	11.7614	0.00%
4	11.8046	11.8046	0.00%
4.2	11.8400	11.8400	0.00%
4.4	11.8690	11.8690	0.00%
4.6	11.8928	11.8928	0.00%
4.8	11.9122	11.9122	0.00%
5	11.9281	11.9281	0.00%
5.2	11.9412	11.9412	0.00%
5.4	11.9518	11.9518	0.00%
5.6	11.9606	11.9606	0.00%
5.8	11.9677	11.9677	0.00%
6	11.9736	11.9736	0.00%
6.2	11.9784	11.9784	0.00%
6.4	11.9823	11.9823	0.00%
6.6	11.9855	11.9855	0.00%
6.8	11.9881	11.9881	0.00%
7	11.9903	11.9903	0.00%
7.2	11.9920	11.9920	0.00%
7.4	11.9935	11.9935	0.00%

7.6	11.9947	11.9947	0.00%
7.8	11.9956	11.9956	0.00%
8	11.9964	11.9964	0.00%
8.2	11.9971	11.9971	0.00%
8.4	11.9976	11.9976	0.00%
8.6	11.9980	11.9980	0.00%
8.8	11.9984	11.9984	0.00%
9	11.9987	11.9987	0.00%
9.2	11.9989	11.9989	0.00%
9.4	11.9991	11.9991	0.00%
9.6	11.9993	11.9993	0.00%
9.8	11.9994	11.9994	0.00%
10	11.9995	11.9995	0.00%
10.2	11.9996	11.9996	0.00%
10.4	11.9997	11.9997	0.00%
10.6	11.9997	11.9997	0.00%
10.8	11.9998	11.9998	0.00%
11	11.9998	11.9998	0.00%
11.2	11.9999	11.9999	0.00%
11.4	11.9999	11.9999	0.00%
11.6	11.9999	11.9999	0.00%
11.8	11.9999	11.9999	0.00%
12	11.9999	11.9999	0.00%
12.2	11.9999	11.9999	0.00%
12.4	12.0000	12.0000	0.00%
12.6	12.0000	12.0000	0.00%
12.8	12.0000	12.0000	0.00%
13	12.0000	12.0000	0.00%
13.2	12.0000	12.0000	0.00%
13.4	12.0000	12.0000	0.00%
13.6	12.0000	12.0000	0.00%
13.8	12.0000	12.0000	0.00%
14	12.0000	12.0000	0.00%
14.2	12.0000	12.0000	0.00%
14.4	12.0000	12.0000	0.00%
14.6	12.0000	12.0000	0.00%
14.8	12.0000	12.0000	0.00%
15	12.0000	12.0000	0.00%
15.2	12.0000	12.0000	0.00%
15.4	12.0000	12.0000	0.00%
15.6	12.0000	12.0000	0.00%
15.8	12.0000	12.0000	0.00%
16	12.0000	12.0000	0.00%
16.2	12.0000	12.0000	0.00%
16.4	12.0000	12.0000	0.00%

16.6	12.0000	12.0000	0.00%
16.8	12.0000	12.0000	0.00%
17	12.0000	12.0000	0.00%
17.2	12.0000	12.0000	0.00%
17.4	12.0000	12.0000	0.00%
17.6	12.0000	12.0000	0.00%
17.8	12.0000	12.0000	0.00%
18	12.0000	12.0000	0.00%
18.2	12.0000	12.0000	0.00%
18.4	12.0000	12.0000	0.00%
18.6	12.0000	12.0000	0.00%
18.8	12.0000	12.0000	0.00%
19	12.0000	12.0000	0.00%
19.2	12.0000	12.0000	0.00%
19.4	12.0000	12.0000	0.00%
19.6	12.0000	12.0000	0.00%
19.8	12.0000	12.0000	0.00%
20	12.0000	12.0000	0.00%

<u>R1 = 10Ω</u>

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	1.0909	1.0909	0.00%
0.2	1.9486	1.9520	0.17%
0.4	2.7513	2.7529	0.06%
0.6	3.4917	3.4923	0.02%
0.8	4.1731	4.1734	0.01%
1	4.7999	4.8001	0.00%
1.2	5.3766	5.3768	0.00%
1.4	5.9070	5.9072	0.00%
1.6	6.3950	6.3952	0.00%
1.8	6.8439	6.8441	0.00%
2	7.2569	7.2571	0.00%
2.2	7.6368	7.6370	0.00%
2.4	7.9862	7.9864	0.00%
2.6	8.3077	8.3079	0.00%
2.8	8.6034	8.6036	0.00%
3	8.8754	8.8757	0.00%
3.2	9.1257	9.1259	0.00%
3.4	9.3559	9.3561	0.00%
3.6	9.5677	9.5679	0.00%
3.8	9.7625	9.7627	0.00%

4	9.9417	9.9419	0.00%
4.2	10.1065	10.1067	0.00%
4.4	10.2582	10.2584	0.00%
4.6	10.3977	10.3979	0.00%
4.8	10.5260	10.5262	0.00%
5	10.6441	10.6442	0.00%
5.2	10.7527	10.7528	0.00%
5.4	10.8526	10.8527	0.00%
5.6	10.9445	10.9446	0.00%
5.8	11.0290	11.0291	0.00%
6	11.1068	11.1069	0.00%
6.2	11.1783	11.1784	0.00%
6.4	11.2441	11.2442	0.00%
6.6	11.3047	11.3048	0.00%
6.8	11.3603	11.3604	0.00%
7	11.4116	11.4117	0.00%
7.2	11.4587	11.4588	0.00%
7.4	11.5021	11.5021	0.00%
7.6	11.5419	11.5420	0.00%
7.8	11.5786	11.5787	0.00%
8	11.6124	11.6124	0.00%
8.2	11.6434	11.6435	0.00%
8.4	11.6720	11.6720	0.00%
8.6	11.6982	11.6983	0.00%
8.8	11.7224	11.7225	0.00%
9	11.7446	11.7447	0.00%
9.2	11.7651	11.7651	0.00%
9.4	11.7839	11.7840	0.00%
9.6	11.8012	11.8013	0.00%
9.8	11.8171	11.8172	0.00%
10	11.8318	11.8318	0.00%
10.2	11.8453	11.8453	0.00%
10.4	11.8577	11.8577	0.00%
10.6	11.8691	11.8691	0.00%
10.8	11.8795	11.8796	0.00%
11	11.8892	11.8892	0.00%
11.2	11.8981	11.8981	0.00%
11.4	11.9062	11.9063	0.00%
11.6	11.9137	11.9138	0.00%
11.8	11.9206	11.9207	0.00%
12	11.9270	11.9270	0.00%
12.2	11.9328	11.9329	0.00%
12.4	11.9382	11.9382	0.00%
12.6	11.9432	11.9432	0.00%
12.8	11.9477	11.9477	0.00%

13	11.9519	11.9519	0.00%
13.2	11.9558	11.9558	0.00%
13.4	11.9593	11.9593	0.00%
13.6	11.9626	11.9626	0.00%
13.8	11.9656	11.9656	0.00%
14	11.9683	11.9683	0.00%
14.2	11.9709	11.9709	0.00%
14.4	11.9732	11.9732	0.00%
14.6	11.9753	11.9753	0.00%
14.8	11.9773	11.9773	0.00%
15	11.9791	11.9791	0.00%
15.2	11.9808	11.9808	0.00%
15.4	11.9823	11.9823	0.00%
15.6	11.9838	11.9838	0.00%
15.8	11.9851	11.9851	0.00%
16	11.9863	11.9863	0.00%
16.2	11.9874	11.9874	0.00%
16.4	11.9884	11.9884	0.00%
16.6	11.9893	11.9893	0.00%
16.8	11.9902	11.9902	0.00%
17	11.9909	11.9909	0.00%
17.2	11.9917	11.9917	0.00%
17.4	11.9923	11.9923	0.00%
17.6	11.9930	11.9930	0.00%
17.8	11.9935	11.9935	0.00%
18	11.9940	11.9940	0.00%
18.2	11.9945	11.9945	0.00%
18.4	11.9950	11.9950	0.00%
18.6	11.9954	11.9954	0.00%
18.8	11.9957	11.9957	0.00%
19	11.9961	11.9961	0.00%
19.2	11.9964	11.9964	0.00%
19.4	11.9967	11.9967	0.00%
19.6	11.9969	11.9969	0.00%
19.8	11.9972	11.9972	0.00%
20	11.9974	11.9974	0.00%

B. Numerical Simulation 2: Comparison of Fourth-order Runge-Kutta method with different step size, h

n = 40 or h = 0.5

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	6.0000	6.0000	0.00%
0.5	16.3100	16.2500	0.37%
1	19.4438	19.4902	0.24%
1.5	15.9570	16.1286	1.08%
2	10.9781	11.1254	1.34%
2.5	8.6979	8.6876	0.12%
3	9.7048	9.5593	1.50%
3.5	11.9767	11.8343	1.19%
4	13.3716	13.3448	0.20%
4.5	13.2249	13.3096	0.64%
5	12.2494	12.3536	0.85%
5.5	11.4772	11.5163	0.34%
6	11.3873	11.3490	0.34%
6.5	11.7762	11.7111	0.55%
7	12.1741	12.1389	0.29%
7.5	12.2894	12.3016	0.10%
8	12.1499	12.1859	0.30%
8.5	11.9568	11.9825	0.22%
9	11.8709	11.8705	0.00%
9.5	11.9123	11.8944	0.15%
10	12.0008	11.9843	0.14%
10.5	12.0540	12.0506	0.03%
11	12.0471	12.0550	0.07%
11.5	12.0089	12.0185	0.08%
12	11.9793	11.9829	0.03%
12.5	11.9763	11.9734	0.02%
13	11.9917	11.9865	0.04%
13.5	12.0070	12.0042	0.02%
14	12.0112	12.0120	0.01%
14.5	12.0057	12.0082	0.02%
15	11.9982	12.0000	0.02%
15.5	11.9950	11.9950	0.00%
16	11.9966	11.9955	0.01%
16.5	12.0001	11.9990	0.01%
17	12.0021	12.0019	0.00%
17.5	12.0018	12.0023	0.00%

18	12.0003	12.0009	0.01%
18.5	11.9992	11.9994	0.00%
19	11.9991	11.9989	0.00%
19.5	11.9997	11.9994	0.00%
20	12.0003	12.0001	0.00%

n = 100 or h = 0.2

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	6.0000	6.0000	0.00%
0.2	10.6791	10.6784	0.01%
0.4	14.6990	14.6984	0.00%
0.6	17.6041	17.6041	0.00%
0.8	19.1806	19.1818	0.01%
1	19.4438	19.4464	0.01%
1.2	18.5942	18.5977	0.02%
1.4	16.9550	16.9590	0.02%
1.6	14.9039	14.9077	0.03%
1.8	12.8090	12.8121	0.02%
2	10.9781	10.9799	0.02%
2.2	9.6254	9.6255	0.00%
2.4	8.8576	8.8561	0.02%
2.6	8.6787	8.6759	0.03%
2.8	9.0075	9.0038	0.04%
3	9.7048	9.7010	0.04%
3.2	10.6042	10.6007	0.03%
3.4	11.5403	11.5376	0.02%
3.6	12.3725	12.3710	0.01%
3.8	13.0005	13.0003	0.00%
4	13.3716	13.3725	0.01%
4.2	13.4791	13.4810	0.01%
4.4	13.3554	13.3579	0.02%
4.6	13.0602	13.0629	0.02%
4.8	12.6669	12.6693	0.02%
5	12.2494	12.2512	0.02%
5.2	11.8719	11.8729	0.01%
5.4	11.5811	11.5814	0.00%
5.6	11.4030	11.4025	0.00%
5.8	11.3426	11.3415	0.01%
6	11.3873	11.3858	0.01%
6.2	11.5115	11.5099	0.01%

6.4	11.6831	11.6816	0.01%
6.6	11.8689	11.8678	0.01%
6.8	12.0398	12.0392	0.01%
7	12.1741	12.1739	0.00%
7.2	12.2591	12.2593	0.00%
7.4	12.2916	12.2922	0.00%
7.6	12.2765	12.2773	0.01%
7.8	12.2245	12.2255	0.01%
8	12.1499	12.1507	0.01%
8.2	12.0673	12.0680	0.01%
8.4	11.9901	11.9905	0.00%
8.6	11.9283	11.9284	0.00%
8.8	11.8879	11.8878	0.00%
9	11.8709	11.8706	0.00%
9.2	11.8755	11.8750	0.00%
9.4	11.8970	11.8965	0.00%
9.6	11.9294	11.9289	0.00%
9.8	11.9660	11.9656	0.00%
10	12.0008	12.0006	0.00%
10.2	12.0292	12.0292	0.00%
10.4	12.0483	12.0484	0.00%
10.6	12.0570	12.0572	0.00%
10.8	12.0560	12.0562	0.00%
11	12.0471	12.0474	0.00%
11.2	12.0331	12.0334	0.00%
11.4	12.0169	12.0171	0.00%
11.6	12.0012	12.0014	0.00%
11.8	11.9882	11.9883	0.00%
12	11.9793	11.9792	0.00%
12.2	11.9749	11.9748	0.00%
12.4	11.9749	11.9748	0.00%
12.6	11.9785	11.9784	0.00%
12.8	11.9845	11.9844	0.00%
13	11.9917	11.9916	0.00%
13.2	11.9987	11.9986	0.00%
13.4	12.0047	12.0046	0.00%
13.6	12.0089	12.0089	0.00%
13.8	12.0110	12.0111	0.00%
14	12.0112	12.0113	0.00%
14.2	12.0098	12.0099	0.00%
14.4	12.0072	12.0073	0.00%
14.6	12.0041	12.0041	0.00%
14.8	12.0009	12.0009	0.00%
15	11.9982	11.9982	0.00%
15.2	11.9962	11.9962	0.00%

15.4	11.9952	11.9951	0.00%
15.6	11.9950	11.9949	0.00%
15.8	11.9955	11.9955	0.00%
16	11.9966	11.9966	0.00%
16.2	11.9980	11.9980	0.00%
16.4	11.9994	11.9994	0.00%
16.6	12.0007	12.0007	0.00%
16.8	12.0016	12.0016	0.00%
17	12.0021	12.0021	0.00%
17.2	12.0022	12.0023	0.00%
17.4	12.0020	12.0020	0.00%
17.6	12.0016	12.0016	0.00%
17.8	12.0009	12.0010	0.00%
18	12.0003	12.0003	0.00%
18.2	11.9998	11.9998	0.00%
18.4	11.9993	11.9993	0.00%
18.6	11.9991	11.9991	0.00%
18.8	11.9990	11.9990	0.00%
19	11.9991	11.9991	0.00%
19.2	11.9993	11.9993	0.00%
19.4	11.9995	11.9995	0.00%
19.6	11.9998	11.9998	0.00%
19.8	12.0001	12.0001	0.00%
20	12.0003	12.0003	0.00%

n = 200 or h = 0.1

Time (s)	Analytical solution	RK-4 method	
	Computed Point	Computed Point	error
0	6.0000	6.0000	0.00%
0.1	8.3844	8.3844	0.00%
0.2	10.6791	10.6791	0.00%
0.3	12.8055	12.8055	0.00%
0.4	14.6990	14.6990	0.00%
0.5	16.3100	16.3100	0.00%
0.6	17.6041	17.6041	0.00%
0.7	18.5624	18.5625	0.00%
0.8	19.1806	19.1807	0.00%
0.9	19.4675	19.4677	0.00%
1	19.4438	19.4440	0.00%
1.1	19.1400	19.1402	0.00%
1.2	18.5942	18.5945	0.00%

1.3	17.8502	17.8504	0.00%
1.4	16.9550	16.9553	0.00%
1.5	15.9570	15.9573	0.00%
1.6	14.9039	14.9041	0.00%
1.7	13.8408	13.8410	0.00%
1.8	12.8090	12.8092	0.00%
1.9	11.8448	11.8449	0.00%
2	10.9781	10.9783	0.00%
2.1	10.2328	10.2328	0.00%
2.2	9.6254	9.6254	0.00%
2.3	9.1659	9.1658	0.00%
2.4	8.8576	8.8575	0.00%
2.5	8.6979	8.6978	0.00%
2.6	8.6787	8.6785	0.00%
2.7	8.7873	8.7871	0.00%
2.8	9.0075	9.0072	0.00%
2.9	9.3202	9.3199	0.00%
3	9.7048	9.7046	0.00%
3.1	10.1400	10.1397	0.00%
3.2	10.6042	10.6040	0.00%
3.3	11.0773	11.0771	0.00%
3.4	11.5403	11.5401	0.00%
3.5	11.9767	11.9766	0.00%
3.6	12.3725	12.3724	0.00%
3.7	12.7165	12.7164	0.00%
3.8	13.0005	13.0005	0.00%
3.9	13.2196	13.2196	0.00%
4	13.3716	13.3716	0.00%
4.1	13.4570	13.4571	0.00%
4.2	13.4791	13.4792	0.00%
4.3	13.4429	13.4430	0.00%
4.4	13.3554	13.3556	0.00%
4.5	13.2249	13.2250	0.00%
4.6	13.0602	13.0604	0.00%
4.7	12.8711	12.8713	0.00%
4.8	12.6669	12.6670	0.00%
4.9	12.4568	12.4569	0.00%
5	12.2494	12.2495	0.00%
5.1	12.0522	12.0523	0.00%
5.2	11.8719	11.8719	0.00%
5.3	11.7135	11.7136	0.00%
5.4	11.5811	11.5811	0.00%
5.5	11.4772	11.4772	0.00%
5.6	11.4030	11.4030	0.00%
5.7	11.3585	11.3585	0.00%

5.8	11.3426	11.3426	0.00%
5.9	11.3532	11.3531	0.00%
6	11.3873	11.3872	0.00%
6.1	11.4413	11.4412	0.00%
6.2	11.5115	11.5114	0.00%
6.3	11.5935	11.5934	0.00%
6.4	11.6831	11.6830	0.00%
6.5	11.7762	11.7761	0.00%
6.6	11.8689	11.8688	0.00%
6.7	11.9578	11.9577	0.00%
6.8	12.0398	12.0398	0.00%
6.9	12.1125	12.1125	0.00%
7	12.1741	12.1741	0.00%
7.1	12.2231	12.2231	0.00%
7.2	12.2591	12.2591	0.00%
7.3	12.2817	12.2818	0.00%
7.4	12.2916	12.2916	0.00%
7.5	12.2894	12.2895	0.00%
7.6	12.2765	12.2765	0.00%
7.7	12.2543	12.2543	0.00%
7.8	12.2245	12.2246	0.00%
7.9	12.1891	12.1892	0.00%
8	12.1499	12.1500	0.00%
8.1	12.1087	12.1088	0.00%
8.2	12.0673	12.0674	0.00%
8.3	12.0273	12.0274	0.00%
8.4	11.9901	11.9901	0.00%
8.5	11.9568	11.9568	0.00%
8.6	11.9283	11.9283	0.00%
8.7	11.9052	11.9052	0.00%
8.8	11.8879	11.8879	0.00%
8.9	11.8766	11.8765	0.00%
9	11.8709	11.8709	0.00%
9.1	11.8707	11.8707	0.00%
9.2	11.8755	11.8755	0.00%
9.3	11.8845	11.8845	0.00%
9.4	11.8970	11.8970	0.00%
9.5	11.9123	11.9122	0.00%
9.6	11.9294	11.9294	0.00%
9.7	11.9476	11.9475	0.00%
9.8	11.9660	11.9660	0.00%
9.9	11.9840	11.9840	0.00%
10	12.0008	12.0008	0.00%
10.1	12.0161	12.0161	0.00%
10.2	12.0292	12.0292	0.00%

10.3	12.0400	12.0400	0.00%
10.4	12.0483	12.0483	0.00%
10.5	12.0540	12.0540	0.00%
10.6	12.0570	12.0570	0.00%
10.7	12.0576	12.0576	0.00%
10.8	12.0560	12.0560	0.00%
10.9	12.0524	12.0524	0.00%
11	12.0471	12.0471	0.00%
11.1	12.0406	12.0406	0.00%
11.2	12.0331	12.0331	0.00%
11.3	12.0251	12.0251	0.00%
11.4	12.0169	12.0169	0.00%
11.5	12.0089	12.0089	0.00%
11.6	12.0012	12.0013	0.00%
11.7	11.9943	11.9943	0.00%
11.8	11.9882	11.9882	0.00%
11.9	11.9832	11.9832	0.00%
12	11.9793	11.9793	0.00%
12.1	11.9765	11.9765	0.00%
12.2	11.9749	11.9749	0.00%
12.3	11.9744	11.9744	0.00%
12.4	11.9749	11.9749	0.00%
12.5	11.9763	11.9763	0.00%
12.6	11.9785	11.9785	0.00%
12.7	11.9813	11.9813	0.00%
12.8	11.9845	11.9845	0.00%
12.9	11.9880	11.9880	0.00%
13	11.9917	11.9917	0.00%
13.1	11.9953	11.9953	0.00%
13.2	11.9987	11.9987	0.00%
13.3	12.0019	12.0019	0.00%
13.4	12.0047	12.0047	0.00%
13.5	12.0070	12.0070	0.00%
13.6	12.0089	12.0089	0.00%
13.7	12.0102	12.0102	0.00%
13.8	12.0110	12.0111	0.00%
13.9	12.0114	12.0114	0.00%
14	12.0112	12.0113	0.00%
14.1	12.0107	12.0107	0.00%
14.2	12.0098	12.0098	0.00%
14.3	12.0086	12.0086	0.00%
14.4	12.0072	12.0072	0.00%
14.5	12.0057	12.0057	0.00%
14.6	12.0041	12.0041	0.00%
14.7	12.0025	12.0025	0.00%

14.8	12.0009	12.0009	0.00%
14.9	11.9995	11.9995	0.00%
15	11.9982	11.9982	0.00%
15.1	11.9971	11.9971	0.00%
15.2	11.9962	11.9962	0.00%
15.3	11.9956	11.9956	0.00%
15.4	11.9952	11.9952	0.00%
15.5	11.9950	11.9950	0.00%
15.6	11.9950	11.9950	0.00%
15.7	11.9952	11.9952	0.00%
15.8	11.9955	11.9955	0.00%
15.9	11.9960	11.9960	0.00%
16	11.9966	11.9966	0.00%
16.1	11.9973	11.9973	0.00%
16.2	11.9980	11.9980	0.00%
16.3	11.9988	11.9988	0.00%
16.4	11.9994	11.9994	0.00%
16.5	12.0001	12.0001	0.00%
16.6	12.0007	12.0007	0.00%
16.7	12.0012	12.0012	0.00%
16.8	12.0016	12.0016	0.00%
16.9	12.0019	12.0019	0.00%
17	12.0021	12.0021	0.00%
17.1	12.0022	12.0022	0.00%
17.2	12.0022	12.0022	0.00%
17.3	12.0022	12.0022	0.00%
17.4	12.0020	12.0020	0.00%
17.5	12.0018	12.0018	0.00%
17.6	12.0016	12.0016	0.00%
17.7	12.0013	12.0013	0.00%
17.8	12.0009	12.0009	0.00%
17.9	12.0006	12.0006	0.00%
18	12.0003	12.0003	0.00%
18.1	12.0000	12.0000	0.00%
18.2	11.9998	11.9998	0.00%
18.3	11.9995	11.9995	0.00%
18.4	11.9993	11.9993	0.00%
18.5	11.9992	11.9992	0.00%
18.6	11.9991	11.9991	0.00%
18.7	11.9990	11.9990	0.00%
18.8	11.9990	11.9990	0.00%
18.9	11.9990	11.9990	0.00%
19	11.9991	11.9991	0.00%
19.1	11.9992	11.9992	0.00%
19.2	11.9993	11.9993	0.00%

19.3	11.9994	11.9994	0.00%
19.4	11.9995	11.9995	0.00%
19.5	11.9997	11.9997	0.00%
19.6	11.9998	11.9998	0.00%
19.7	12.0000	12.0000	0.00%
19.8	12.0001	12.0001	0.00%
19.9	12.0002	12.0002	0.00%
20	12.0003	12.0003	0.00%

APPENDIX D

Gantt Chart for FYP 1

A. PLANNING FOR FYP 1																	
No	TASKS	Timeline															
		September 2016				October 2016				November 2016				December 2016			
		W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12	W13	W14	W15	W16
1.	Selection of Title																
2.	Study on Related Articles																
3.	Objective and Scope																
4.	Literature Review																
5.	Methodology																
6.	Experiments																
7.	Results and Analysis																
8.	Documentation																
9.	Submission of Draft																
10.	FYP 1 Seminar																
11.	Final Report Writing																

Gantt Chart for FYP 2

B. PLANNING FOR FYP 2																	
No	TASKS	Timeline															
		February 2017				March 2017				April 2017				May 2017			
		W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12	W13	W14	W15	W16
1.	Selection of Title																
2.	Study on Related Articles																
3.	Objective and Scope																
4.	Literature Review																
5.	Methodology																
6.	Experiments																
7.	Results and Analysis																
8.	Documentation																
9.	Submission of Draft																
10.	FYP 2 Seminar																
11.	Final Report Writing																